Your name ___________________________

Harvey Mudd College
CS 81 Mid-Term Exam Solutions
Spring semester, 2010

- Please write legibly.
- You may use a single 2-sided 8.5” x 11” reference sheet, but no other reference material.
- Please turn in your reference sheet along with the exam. It will be returned to you.
- The exam has a 75-minute overall time limit. Do the best you can in that time. It is best to spend your time working on problems that pay off in terms of the most points.
- Note that some questions say “prove”, while others say “prove or disprove”, leaving it up to you to decide which is possible.

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1. [10 points]

Prove the two sequents below using the Sequent Calculus. (If unable to do it with the Sequent Calculus, you may use an alternate method for half-credit.) The formulas are repeated at the bottom of the space for your convenience.
Part I:

One of several possible proofs:

$q, r \vdash \neg p, q$ *Ax*

$p \vdash \neg r, p, q$ *Ax* $\qquad r \vdash \neg q, p, q$ $\vdash\neg$

$r, p \vdash r$ *Ax* $\qquad p \vdash \neg r, (p \vee q)$ $\vdash\vee$ $\qquad r \vdash \neg q, (p \vee q)$ $\vdash\vee$

$p \vdash r, \neg r$ $\vdash\neg$ $\qquad r \vdash r, \neg q$ *Ax* $\qquad \neg(p \vee q), p \vdash \neg r$ $\qquad \neg(p \vee q), r \vdash \neg q$ $\neg\vdash$

$\vdash r, (p\rightarrow\neg r)$ $\vdash\rightarrow$ $\qquad \vdash r, (r\rightarrow\neg q)$ $\vdash\rightarrow$ $\qquad \neg(p\vee q) \vdash (p\rightarrow\neg r)$ $\vdash\rightarrow$ $\qquad \neg(p\vee q) \vdash (r\rightarrow\neg q)$ $\vdash\rightarrow$

$\vdash r, (p\rightarrow\neg r) \wedge (r\rightarrow\neg q)$ $\vdash\wedge$ $\qquad \neg(p \vee q) \vdash (p\rightarrow\neg r) \wedge (r\rightarrow\neg q)$ $\vdash\wedge$

$r \rightarrow \neg(p \vee q) \vdash (p\rightarrow\neg r) \wedge (r\rightarrow\neg q)$ $\vdash\rightarrow$

Part II (using Sequent Calculus): (∃x R(x)) → S(a) ⊢ (∀y ¬R(y)) ∨ (∃z S(z))

One of several possible proofs:

R(y_0) ⊢ R(y_0), (∃z S(z)) *Axiom*

R(y_0) ⊢ (∃x R(x)), (∃z S(z)) ⊢∃

⊢ (∃x R(x)), ¬R(y_0), (∃z S(z)) ⊢¬ S(a) ⊢ (∀y ¬R(y)), S(a) *Axiom*

⊢ (∃x R(x)), (∀y ¬R(y)), (∃z S(z)) ⊢ ∀ S(a) ⊢ (∀y ¬R(y)), (∃z S(z)) ⊢∃

(∃x R(x)) → S(a) ⊢ (∀y ¬R(y)), (∃z S(z)) → ⊢

(∃x R(x)) → S(a) ⊢ (∀y ¬R(y)) ∨ (∃z S(z)) ⊢ ∨

2. [10 points] Prove or disprove each sequent, using any method you wish.

Part I:

$\exists x\ P(x),\ \exists x\ (P(x) \rightarrow Q(x)) \vdash \exists x\ Q(x)$

This sequent is not valid.

Counterexample interpretation I: $\Delta = \{1, 2\}$, $\mu(P) = \{(1)\}$, $\mu(Q) = \varnothing$.

Here $I(\exists x\ P(x)) = T$, since $\mu(P)(1)$. Also $I(\exists x\ (P(x) \rightarrow Q(x))) = T$, since $I(P(2) \rightarrow Q(2)) = T$. However, $I(\exists x\ Q(x)) = F$.

Part II:

$\forall x\ \exists y\ P(x, y),\ \forall y\ \exists x\ P(x, y) \vdash \exists x\ P(x, x)$

This sequent is not valid.

Counterexample interpretation I: $\Delta = \{1, 2\}$, $\mu(P) = \{(1, 2), (2, 1)\}$.

Here $I(\forall x\ \exists y\ P(x, y)) = T$, $I(\forall y\ \exists x\ P(x, y)) = T$, but $I(\exists x\ P(x, x)) = F$.

3. [20 points] Prove formula d. below, using natural deduction and the Peano axioms (a, b, c). Recall that $\varphi[t/x]$, where t is a term, means φ with t replacing each free occurrence of x.

   a. $\forall x \neg(s(x) = 0)$
   b. $\forall x \forall y\ (s(x) = s(y) \rightarrow x = y)$
   c. [Induction axioms] For any formula φ: $(\varphi[0/x] \wedge \forall x\ (\varphi \rightarrow \varphi[s(x)/x])) \rightarrow \forall x\ \varphi$

   d. To be proved: $\forall x \neg(s(x) = x)$.

Proof by induction, where φ is $\neg(s(x) = x)$.

1.	$\forall x \neg(s(x) = 0)$				a.
2.	$\neg(s(0) = 0)$				1, $\forall$E
3.		x_0			Fresh variable
4.			$\neg(s(x_0) = x_0)$		Assumption
5.				$s(s(x_0)) = s(x_0)$	Assumption
6.				$\forall y\ (s(s(x_0)) = s(y) \rightarrow s(x_0) = y)$	b, $\forall$E (sub s(x) for x)
7.				$s(s(x_0)) = s(x_0) \rightarrow s(x_0) = x_0$	6, $\forall$E (sub x for y)
8.				$s(x_0) = x_0$	5, 7, $\rightarrow$E
9.				$\bot$	4, 8, $\neg$E
10.			$\neg(s(s(x_0)) = s(x_0))$		5-9, $\neg$I
11.		$\neg(s(x_0) = x_0) \rightarrow \neg(s(s(x_0)) = s(x_0))$			4-10, $\rightarrow$I
12.	$\forall x\ (\neg(s(x) = x) \rightarrow \neg(s(s(x)) = s(x)))$				3-11, $\forall$I
13.	$\neg(s(0) = 0) \wedge \forall x\ (\neg(s(x) = x) \rightarrow \neg(s(s(x)) = s(x)))$				2, 12, $\wedge$I
14.	$(\neg(s(0) = 0) \wedge \forall x\ (\neg(s(x) = x) \rightarrow \neg(s(s(x)) = s(x)))) \rightarrow \forall x \neg(s(x) = x))$				c, with φ as $\neg(s(x) = x)$
15.	$\forall x \neg(s(x) = x)$				13, 14, $\rightarrow$E

Note that the induction step, line 11, is an instance of the contraposition of the body of b.

4. [15 points]

Carefully prove using natural deduction:

$\exists x\, P(x) \wedge \exists x\, Q(x) \vdash \exists x\, \exists y\, (P(x) \wedge Q(y))$

Proof:

1.	$\exists x\, P(x) \wedge \exists x\, Q(x)$			Premise
2.	$\exists x\, P(x)$			1, ∧E
3.	$\exists x\, Q(x)$			1, ∧E
4.		$P(x_0)$		Assumption
5.			$Q(y_0)$	Assumption
6.			$P(x_0) \wedge Q(y_0)$	4, 5, ∧I
7.			$\exists y\, (P(x_0) \wedge Q(y))$	6, ∃I
8.			$\exists x\, \exists y\, (P(x) \wedge Q(y))$	7, ∃I
9.		$\exists x\, \exists y\, (P(x) \wedge Q(y))$		3, 5-8, ∃E
10.	$\exists x\, \exists y\, (P(x) \wedge Q(y))$			2, 4-9, ∃E

1. [15 points]
Translate each of Parts I and II below into one or more formulas, then prove or disprove each part, using any method (the premises of both parts are the same, so you only have to provide one translation of them). Provide any insightful comments that connect the two parts.

Part I:

a. John loves a person iff that person does not love him or herself.
b. John is a person and Mary is a person
c. Mary loves herself.

Therefore d. John does not love Mary.

Translation (For simplicity, everything mentioned is a person.):

a1. ∀x (loves(John, x) → ¬loves(x, x))
a2. ∀x (¬loves(x, x) → loves(John, x))
d. ¬loves(John, Mary)

Part II:

a. John loves a person iff that person does not love him or herself.
b. John is a person and Mary is a person.
c. Mary loves herself.

Therefore e. John loves Mary.

Comment: Premise a. is inconsistent (⊥ can be derived from it), as shown below. Therefore every formula follows from that premise, even mutually contradictory ones, such as d and e above.

A natural deduction proof:

1.	loves(John, John) ∨ ¬loves(John, John)		LEM
2.		loves(John, John)	Assumption
3.		loves(John, John) → ¬loves(John, John)	a1, ∀E
4.		¬loves(John, John)	2, 3, →E
5.		⊥	2, 4, ¬E
6.		¬loves(John, John)	Assumption
7.		¬loves(John, John) → loves(John, John)	a2, ∀E
8.		loves(John, John)	6, 7, →E
9.		⊥	6, 8, ¬E
10.	⊥		1, 2-5, 6-9, ∨E
11.	loves(John, Mary)		10, ⊥E
12.	¬loves(John, Mary)		10, ⊥E

A resolution proof that just a. is unsatisfiable, starting with clausal forms:

a1. ¬loves(John, x) ∨ ¬loves(x, x)
a2. loves(x, x) ∨ loves(John, x)

These unify, using x = John in the first clause. After substitution, we have ¬loves(John, John) with loves(John, John), which gives ⊥.

6. [20 points]

Using Hoare Logic, and any relevant algebraic identities (which need not be proved), prove the partial correctness of the following triple:

$\{n \geq 0\}$

___________________ $\{n \geq 0 \wedge 1 = 2^0\}$

`x := 0;`

___________________ $\{n \geq x \wedge 1 = 2^x\}$

`y := 1;`

___________________ $\{n \geq x \wedge y = 2^x\}$

`while x < n do`

__________ $\{x < n \wedge n \geq x \wedge y = 2^x\}$

__________ $\{n \geq x+1 \wedge y+y = 2^{x+1}\}$

`y := y + y;`

__________ $\{n \geq x+1 \wedge y = 2^{x+1}\}$

`x := x + 1`

__________ $\{n \geq x \wedge y = 2^x\}$

`od`

_______________ $\{\neg(x < n) \wedge n \geq x \wedge y = 2^x\}$

$\{y = 2^n\}$

Suggestion: Show the assertions between each line off to the right (where the lines are), then establish the relevant triples.

The invariant for the while loop is: $\{n \geq x \wedge y = 2^x\}$

The Hoare triples are:

$\{n \geq 0 \wedge 1 = 2^0\}$ `x := 0` $\{n \geq x \wedge 1 = 2^x\}$ *assignment*

$\{n \geq x \wedge 1 = 2^x\}$ `y := 1` $\{n \geq x \wedge y = 2^x\}$ *assignment*

$\{n \geq x+1 \wedge y+y = 2^{x+1}\}$ `y := y + y` $\{n \geq x+1 \wedge y = 2^{x+1}\}$ *assignment*

$\{n \geq x+1 \wedge y = 2^{x+1}\}$ `x := x + 1` $\{n \geq x \wedge y = 2^x\}$ *assignment*

$\{n \geq 0 \wedge 1 = 2^0\}$ `x := 0; y := 1` $\{n \geq x \wedge y = 2^x\}$ *composition*

$\{x < n \wedge n \geq x \wedge y = 2^x\}$ `y:=y+y; x:=x+1` $\{n \geq x \wedge y = 2^x\}$ *composition*

$\{n \geq x \wedge y = 2^x\}$ `while … od` $\{\neg(x < n) \wedge n \geq x \wedge y = 2^x\}$ *while*

$\{n \geq 0 \wedge 1 = 2^0\}$ *entire program* $\{y = 2^n\}$ *composition*

The identities used are:

$1 = 2^0$

$y = 2^x \rightarrow y+y = 2^{x+1}$

$x < n \rightarrow n \geq x+1$

$\neg(x < n) \wedge n \geq x \rightarrow x = n$

$(y = 2^x \wedge n = x) \rightarrow y = 2^n$

7. [10 points]

Using the clauses below and resolution, prove that 2+2 = 4 and 2×2 = 4, where number 2 is represented as `s(s(0))`, and the number 4 is represented as `s(s(s(s(0))))`, where `s` is the successor function.

a(x, y, z) means that z is the sum of x and y, and m(x, y, z) means that z is the product of x and y. The following clauses derive directly from the number theory axioms.

1. a(0, x, x).	1. a(0, x, x)
2. a(x, y, z) → a(s(x), y, s(z)).	2. ¬a(x, y, z) ∨ a(s(x), y, s(z))
3. m(0, x, 0).	3. m(0, x, 0)
4. (m(x, y, z) ∧ a(y, z, w)) → m(s(x), y, w).	4. ¬m(x, y, z) ∨ ¬a(y, z, w)) ∨ m(s(x), y, w)

2+2=4: Conclusion: a(s(s(0)), s(s(0)), s(s(s(s(0)))))

5. ¬a(s(s(0)), s(s(0)), s(s(s(s(0))))).	¬ conclusion
6. ¬a(s(0), s(s(0)), s(s(s(0)))).	2, 5 resolution
7. ¬a(0, s(s(0)), s(s(0))).	2, 6, resolution
8. ⊥	1, 7, resolution

2×2 = 4: Conclusion: m(s(s(0)), s(s(0)), s(s(s(s(0))))).

Starting anew from 1-4:

9. ¬m(s(s(0)), s(s(0)), s(s(s(s(0))))).	¬ conclusion
10. ¬m(s(0), s(s(0)), z) ∨ ¬a(s(s(0)), z, s(s(s(s(0))))).	4, 9 resolution
11. ¬m(0, s(s(0)), zz) ∨ ¬a(s(s(0)), zz, z) ∨ ¬a(s(s(0)), z, s(s(s(s(0))))).	4, 10 resolution
12. ¬a(s(s(0)), 0, z) ∨ ¬a(s(s(0)), z, s(s(s(s(0))))).	3, 11 resolution
13. ¬a(s(0), 0, z) ∨ ¬a(s(s(0)), s(z), s(s(s(s(0))))).	2, 12 resolution
14. a(0, 0, z) ∨ ¬a(s(s(0)), s(s(z)), s(s(s(s(0))))).	2, 13 resolution
15. ¬a(s(s(0)), s(s(0)), s(s(s(s(0))))).	1, 14 resolution
16. ¬a(s(0), s(s(0)), s(s(s(0)))).	2, 15 resolution
17. ¬a(0, s(s(0)), s(s(0))).	2, 16 resolution
18. ⊥	1, 17 resolution

Note the linearity of the resolution steps.