Resolution Theorem Proving, Part 3

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The Need for “Factoring”

- The simple form of resolution (called “binary resolution”) used so far is not quite enough for full generality.
- Consider these clauses:
  - $p(X) \vee p(Y)$
  - $\neg p(U) \vee \neg p(V)$
- There are four ways to resolve these two clauses (e.g. $\{X \leftarrow U\}$). However, no resolvent introduces anything new. In order to make progress, we need to “factor” the clauses.

Factoring

- If there are two are more literals *in the same clause* that unify, then the result of reducing the clause after applying the mgu is called a **factor** of the clause.

- Example:
  - In clause $p(X) \vee p(Y)$, $\{X \leftarrow Y\}$ unifies the two literals.
  - The reduced form is $p(Y)$, which is a factor of the original clause.
  - Evidently this clause could be **replaced** with its factor. However, **this will not always be the case.**

Factoring: Another Example

- In clause

  $p(X) \vee p(f(Y)) \vee p(f(g(Z))) \vee q(Y)$

  $\{X \leftarrow f(g(Z)), Y \leftarrow g(Z)\}$ unifies the first three literals.

- The corresponding factor is:

  $p(f(g(Z))) \vee q(g(Z))$

- The factor is, however, **less general**, so we **cannot replace** the original clause with the factor.

General Resolution of Two Clauses

- Two clauses resolve if:
  - They have a binary resolvent (the simplest kind of resolution, without factoring).
    or
  - One clause and a factor of the other have a binary resolvent.
    or
  - There are factors of the two clauses that have a binary resolvent.

- Since a clause is trivially a factor of itself, we could get by with just the third statement above.

Full Resolution Example

- Clauses:
  - $p(X, Y) \vee p(Y, X)$
  - $\neg p(U, V) \vee \neg p(V, U)$
- Factors:
  - $p(X, X)$
  - $\neg p(U, U)$
- Resolvent:
  - $\perp$

Full Resolution Example

- Clauses:
  1. $p(X, Y) \vee q(X, Y)$
  2. $p(U, V) \vee \neg q(U, g(W))$
  3. $\neg p(f(R), S) \vee \neg p(f(S), g(T))$
- Resolvents:
  4. $p(U, V) \vee p(U, g(W))$ 1, 2 with $\{X \leftarrow U, Y \leftarrow g(W)\}$
  5. $\perp$ 3, 4 with $\{U \leftarrow f(g(T)), R \leftarrow g(T),$
     $S \leftarrow g(T), V \leftarrow g(T), W \leftarrow T$

Subsumption

- A clause C **subsumes** a clause D if there is a substitution θ such that $C\theta \subseteq D$, where we interpret the clauses as **sets** of their literals.

- If a clause D in a set of clauses is subsumed by another clause C ***within the set***, then we can delete D from the set without affecting the case of whether the empty clause $\perp$ is derivable.

Subsumption Examples

- $P(X)$ subsumes $P(X) \vee Q(Y)$
  (by the empty substitution $\{\}$).

- $\neg P(X) \vee Q(f(X))$ subsumes
  $\neg P(Z) \vee \neg P(h(Y)) \vee Q(f(h(Y)))$
  (by the substitution $\{X \leftarrow h(Y), Z \leftarrow h(Y)\}$).

Answer Extraction

- Resolution is not just for proving theorems anymore.
- Resolution can be used for extracting answers from a database, knowledge base, or reasoning system.

From Yes-No to Answer Terms

- Consider the clause set:
  - $\neg$man(X) $\vee$ mortal(X)
  - man(socrates)
  - $\neg$mortal(socrates)
- Obviously this set is unsatisfiable, and we can obtain a proof by resolution.
- But what if we drop the third clause. The first two clauses are satisfiable, and can be thought of as a "knowledge base".
- We can ask a question of the knowledge base:

  Name someone who is mortal.

Asking Questions

- To find an individual for X that satisfies a criterion p(…X…), add to the set of clauses the clause:

  $\neg p(\ldots X \ldots) \vee answer(X)$

- Then conduct resolution as before, but stop when there is a clause containing only answer literals.

Example: Who is mortal?

1. $\neg man(X) \vee mortal(X)$
2. $man(socrates)$
3. $\neg mortal(Y) \vee answer(Y)$
4. $mortal(socrates)$ resolution 1, 2
5. $answer(socrates)$ resolution 2, 3

Example: Who is Caroline's Grandfather?

1.	$\neg$father(X, Y) $\vee$ parent(X, Y)	
2.	$\neg$father(X, Y) $\vee$ $\neg$parent(Y, Z) $\vee$ grandfather(X, Z)	
3.	father(joe, john)	
4.	father(john, caroline)	
5.	$\neg$grandfather(X, caroline) $\vee$ answer(X)	
6.	$\neg$father(X, Y) $\vee$ $\neg$parent(Y, caroline) $\vee$ answer(X)	2, 5
7.	$\neg$father(X, Y) $\vee$ $\neg$father(Y, caroline) $\vee$ answer(X)	1, 6
8.	$\neg$father(X, john) $\vee$ answer(X)	4, 7
9.	answer(joe)	3, 8

Answer Extraction in Otter

- Normally Otter searches for the null clause and stops when and if it has produced it.

- If the special literal

  $answer(X)

  appears in a clause, Otter will stop when it finds a clause containing only literals containing $answer.

Grandfather Example in Otter

```
-father(x, y) | parent(x, y).

-father(x, y) | -parent(y, z) | grandfather(x, z).

father(joe, john).

father(john, caroline).

-grandfather(x, caroline) | $answer(x).
```

```
1 [] -father(x,y)|parent(x,y).
2 [] -father(x,y)| -parent(y,z)|grandfather(x,z).
3 [] -grandfather(x,caroline)|$answer(x).
4 [] father(joe,john).
5 [] father(john,caroline).
7 [hyper,5,1] parent(john,caroline).
8 [hyper,7,2,4] grandfather(joe,caroline).
9 [binary,8.1,3.1] $answer(joe).
```

“Logic” Puzzles: Example

```
% Professors Dodds, Stone, and Thom go to their favorite bars for beer.
% Each prof prefers a different beer (one of Anchor, Bud, and Miller)
% and frequents a different bar (one of Alice's, Harry's, or Joe's).
% Each bar serves a unique beer.
%
% Professor Stone prefers Bud.                              (Clue 1)
%
% Professor Thom doesn't prefer Miller.                     (Clue 2)
%
% The prof who prefers Miller frequents Alice's bar.        (Clue 3)
%
% The prof who prefers Anchor does not frequent Joe's.      (Clue 4)
%
%
% Which bar does each prof frequent and what beer does each prefer?
```

Solving

- Determine clause form from clues.
- We will use Otter form, so that Otter can try to solve the puzzle.
- Note: This may look really simple, but it is not always easy to get right.

Define clauses for clues

- prefer(Stone, Bud). % Clue 1
- -prefer(Thom, Miller). % Clue 2
- -prefer(x, Miller) | frequent(x, Alice). % Clue 3
- -prefer(x, Anchor) | -frequent(x, Joe). % Clue 4

Idenfity Individuals in Various Categories

- prof(Dodds).
  prof(Stone).
  prof(Thom).

- beer(Anchor).
  beer(Bud).
  beer(Miller).

- bar(Alice).
  bar(Harry).
  bar(Joe).

Distribution Requirements

- % Every bar is frequented by some prof.
- -bar(y) | frequent(Dodds, y) | frequent(Stone, y) | frequent(Thom, y).
- % Every beer is preferred by some prof.
- -beer(y) | prefer(Dodds, y) | prefer(Stone, y) | prefer(Thom, y).

Uniqueness Requirements

- % Each bar serves a unique beer.
- -serves(x, y) | -serves(x, z) | y = z.

- % Each prof prefers a unique beer.
- -prefer(x, y) | -prefer(x, z) | y = z.

- % Each prof frequents a unique bar.
- -frequent(x, y) | -frequent(x, z) | y = z.

Answer Clause

Which bars are frequented, and which beers preferred, by which professors?

-frequent(Dodds, x) | -frequent(Stone, y) | -frequent(Thom, z)

| -prefer(Dodds, u) | - prefer(Stone, v) | -prefer(Thom, w)

| $answer([Dodds, x, u], [Stone, y, v], [Thom, z, w]).

Otter Solution

```
$answer([Dodds,Alice,Miller],
        [Stone,Joe,Bud],
        [Thom,Harry,Anchor]).
```

What if Solution not Unique?

Try removing one or more clues.

Note that Otter will give a *disjunctive* solution.

```
  $answer([Dodds,Harry,Anchor],
          [Stone,Joe,Bud],
          [Thom,Alice,Miller])
| $answer([Dodds,Alice,Miller],
          [Stone,Joe,Bud],
          [Thom,Harry,Anchor]).
```

What if No Solution?

If there is no refutation,

Otter will run out of clauses to create,

or run forever.

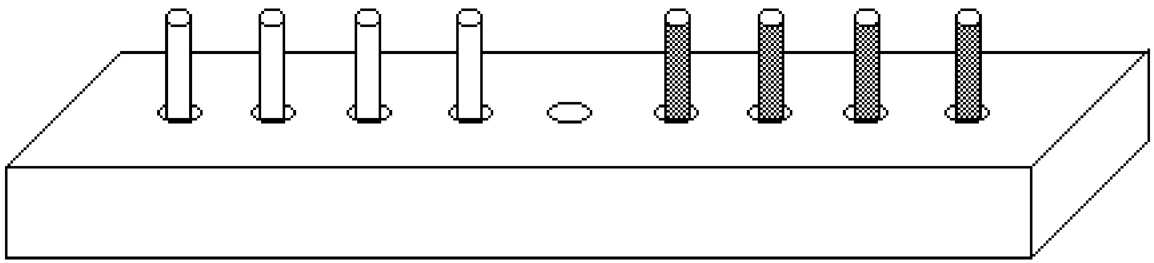
Set of Support (sos) Strategy

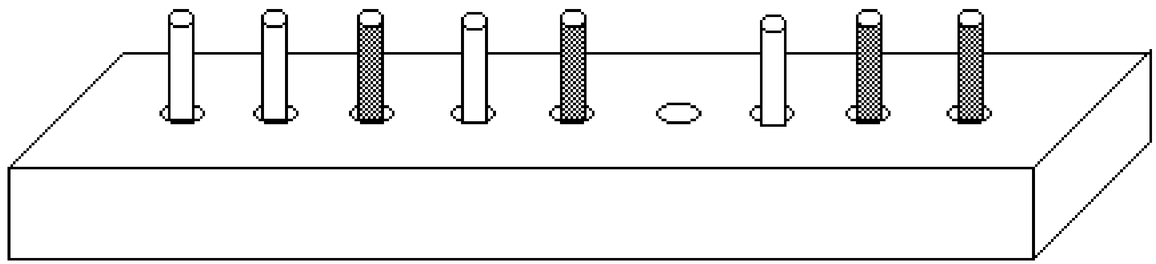
- A typical clause set to be refuted will involve:
  - A set of clauses known, or thought to be **mutually consistent** (satisfiable), e.g. derived from axioms.
  - A single clause which is derived from the negation of the "theorem" to be proved.
- The sos strategy entails always picking one clause for resolution from the sos, and others from outside the sos.
- Resolvents are added to the sos.
- This is a complete strategy and is the one used by Otter.

Motion Puzzles and Games

- Moves in a motion puzzle or game can often be encoded as logic.
- Resolution can be used to find a solving or winning sequence of moves.

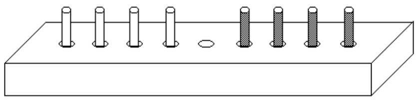
Example: Linear Peg Solitaire

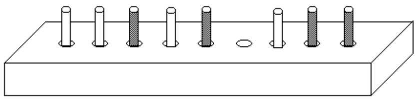




Linear Peg Solitaire Explanation

- Pegs of two colors are shown in their home positions at the top.
- The objective is to completely reverse the pegs, so that each peg's original home is occupied by a peg of the opposite color.
- Allowable actions:
  - Move: A peg can be moved toward the opposite side by moving into an adjacent empty hole.
  - Jump: A peg can jump toward the opposite side over a peg of the opposite color, provided that there is a hole to receive the jumping peg.





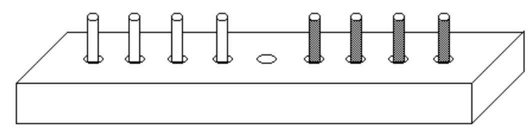
General Form of the Puzzle

- Versions of the puzzle exists for 2n pegs (n of each color) and 2n+1 holes.

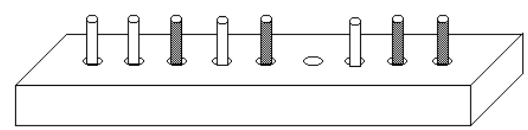
- Ideally, each version can be solved.

Peg Game Formulation

- Represent the **state** of the game with two terms.
- Say the pegs are **w** for white, **r** for red.
- Represents the pegs **away from the hole** in either direction as a composition of function symbols.
- The initial state shown is:
  s(w(w(w(w(c)))), r(r(r(r(c)))



- The second state shown is:
  s(r(w(r(w(w(c))))), w(r(r(c))



- c is a dummy constant symbol

Formulating Moves

- Simple moves (non-jump):
  - move(s(w(X), Y), s(X, w(Y))) (wm)
  - move(s(X, r(Y)), s(r(X), Y)) (rm)

- Jump moves:
  - move(s(r(w(X)), Y), s(X, r(w(Y))) (wj)
  - move(s(X), w(r(Y))), s(w(r(X)), Y)) (rj)

Formulating Reachability

- Initial state:
  reachable(w(w(w(w(c)))), r(r(r(r(c)))))

- State change:
  ¬reachable(X) ∨ ¬move(X, Y) ∨ reachable(Y)

- Final state:
  ¬reachable(r(r(r(r(c)))), w(w(w(w(c)))))

Otter Formulation

```
move(s(w(x), y), s(x, w(y))).
move(s(x, r(y)), s(r(x), y)).

move(s(r(w(x)), y), s(x, r(w(y)))).
move(s(x, w(r(y))), s(w(r(x)), y)).

reachable(s(w(w(w(w(c)))), r(r(r(r(c)))))).

-reachable(x) | -move(x, y) | reachable(y).

-reachable(s(r(r(r(r(c)))), w(w(w(w(c)))))).
```

Otter proof for 2 pegs of each color

```
1 [] -reachable(x)| -move(x,y)|reachable(y).
2 [] -reachable(s(r(r(c)),w(w(c)))).
3 [] move(s(w(x),y),s(x,w(y))).
4 [] move(s(x,r(y)),s(r(x),y)).
5 [] move(s(r(w(x)),y),s(x,r(w(y)))).
6 [] move(s(x,w(r(y))),s(w(r(x)),y)).
7 [] reachable(s(w(w(c)),r(r(c)))).
10 [hyper,4,1,7] reachable(s(r(w(w(c))),r(c))).
11 [hyper,10,1,5] reachable(s(w(c),r(w(r(c))))).
14 [hyper,11,1,3] reachable(s(c,w(r(w(r(c)))))).
18 [hyper,14,1,6] reachable(s(w(r(c)),w(r(c)))).
22 [hyper,18,1,6] reachable(s(w(r(w(r(c)))),c)).
24 [hyper,22,1,3] reachable(s(r(w(r(c))),w(c))).
26 [hyper,24,1,5] reachable(s(r(c),r(w(w(c))))).
28 [hyper,26,1,4] reachable(s(r(r(c)),w(w(c)))).
29 [binary,28.1,2.1] $F.
```

Otter Proof for 3 pegs of each color

```
7  []              reachable(s(w(w(w(c))),r(r(r(c))))).
8  [hyper,7,1,4]   reachable(s(r(w(w(w(c)))),r(r(c)))).
12 [hyper,5,1,8]   reachable(s(w(w(c)),r(w(r(r(c)))))).
16 [hyper,12,1,3] reachable(s(w(c),w(r(w(r(r(c))))))).
20 [hyper,16,1,6] reachable(s(w(r(w(c))),w(r(r(c))))).
27 [hyper,20,1,6] reachable(s(w(r(w(r(w(c))))),r(c))).
34 [hyper,27,1,4] reachable(s(r(w(r(w(r(w(c)))))),c)).
39 [hyper,34,1,5] reachable(s(r(w(r(w(c)))),r(w(c)))).
44 [hyper,39,1,5] reachable(s(r(w(c)),r(w(r(w(c)))))).
51 [hyper,44,1,5] reachable(s(c,r(w(r(w(r(w(c)))))))).
57 [hyper,51,1,4] reachable(s(r(c),w(r(w(r(w(c))))))).
63 [hyper,57,1,6] reachable(s(w(r(r(c))),w(r(w(c))))).
69 [hyper,63,1,6] reachable(s(w(r(w(r(r(c))))),w(c))).
72 [hyper,69,1,3] reachable(s(r(w(r(r(c)))),w(w(c)))).
75 [hyper,72,1,5] reachable(s(r(r(c)),r(w(w(w(c)))))).
77 [hyper,75,1,4] reachable(s(r(r(r(c))),w(w(w(c))))).
78 [binary,77.1,2.1] $F.
```

Pegs vs. Proof Length (# of Moves)

Pegs of Each Color	Proof Length
1	3
2	8
3	15
4	24
5	35
n	$n^2 + 2n$

Determining the Move Sequence

- The previous proofs only showed that the puzzle could be solved for those variations.
- The actual move sequence would have to be dug out from the proof steps.
- We can modify the rules so that the move sequence is obtained as a **byproduct**.

Determining the Move Sequence

- Use function composition to represent accumulated move sequence.
- Revised rules (4-pegs, where specific):
  - move(s(w(x), y), s(x, w(y)), z, wm(z)).
  - move(s(x, r(y)), s(r(x), y), z, rm(z)).

  - move(s(r(w(x)), y), s(x, r(w(y))), z, wj(z)).
  - move(s(x, w(r(y))), s(w(r(x)), y), z, rj(z)).

  - reachable(s(w(w(w(w(c)))), r(r(r(r(c))))), d).

  - -reachable(x, z) | -move(x, y, z, zz) | reachable(y, zz).

  - -reachable(s(r(r(r(r(c)))), w(w(w(w(c))))), z) | $answer(z).

The Move Sequence is Read Inside-Out:

- For 4 pegs of each color:
  $answer(rm(wj(wm(rj(rj(rm(wj(wj(wj(wm(rj(rj(rj(rj(wm(wj(wj(wj(rm(rj(rj(wm(wj(rm(d))))))))))))))))))))))))).

- The sequence is:
  rm wj wm rj rj rm wj wj wj wm rj rj
  rj rj wm wj wj wj rm rj rj wm wj rm

- (For this puzzle, the move sequence is coincidentally a palindrome.)

Other Notes on Otter

- Otter can preprocess formulas into clauses for you:
  - Quantifiers as input
  - Automatic prenexing and skolemization
- Otter can automatically determine *an* sos.

Example from the Mid-Term

- Given:
- $\forall x\ \exists y\ R(x, y)$
- $\forall x\ \forall y\ \forall z\ ((R(x, y) \wedge R(y, z)) \rightarrow R(x, z))$
- $\forall x\ \forall y\ (R(x, y) \rightarrow R(y, x))$

To derive:
- $\forall x\ R(x, x)$

Otter Input

```
formula_list(usable). % use formula_list rather than list

all x (exists y r(x, y)).

all x all y all z ((r(x, y) & r(y, z)) -> r(x, z)).

all x all y (r(x, y) -> r(y, x)).

-(all x r(x, x)).

end_of_list.
```

Result of Pre-Processing by Otter

```
r(x,$f1(x)).    %Auto-identified as sos by Otter
-r(x,y)| -r(y,z)|r(x,z).
-r(x,y)|r(y,x).
-r($c1,$c1).
```

Note that $ is Otter's way of specifying generated Skolem functions and constants.
The original list was:

```
all x (exists y r(x, y)).

all x all y all z ((r(x, y) & r(y, z)) -> r(x, z)).

all x all y (r(x, y) -> r(y, x)).

-(all x r(x, x)).
```

Otter’s Proof of the Midterm Problem

```
1 [] -r(x,y)| -r(y,z)|r(x,z).
2 [] -r(x,y)|r(y,x).
3 [] -r($c1,$c1).
4 [] r(x,$f1(x)).
5 [hyper,4,2] r($f1(x),x).
8 [hyper,5,1,4] r(x,x).
9 [binary,8.1,3.1] $F.
```

Otter Proof Rules and Nomenclature

- binary: means binary resolution
- factoring: is indicated when used
- hyper: means hyper-resolution: resolving multiple clauses in one step.
- paramodulation: a rule for handling equality
- demodulation: use of user-specified equalities
- Knuth-Bendix: a system for pre-processing equality rules

Otter is Not Prolog

- The syntax is different, although Otter has a “Prolog variables” mode.
- Otter has genuine negation.
- Prolog only has “negation as failure”.
- Prolog relies on the “closed world assumption”.