Computers: What can't they do!
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Part 1: Decision Problems and State Machines and Decision Problems

November 17, 2011

CS 42: Principles and Practice of Computer Science

Let what you say be simply 'Yes' or 'No'; anything more than this comes from evil.

Matthew 5:37 (English Standard Version)

It from bit. Otherwise put, every it—every particle, every field of force, even the space-time continuum itself—derives its function, its meaning, its very existence entirely …[from] answers to yes or no questions, binary choices, bits.

John Archibald Wheeler

Theocomp

Computability: What can and can't we compute?

Complexity: How fast can we compute it?

Computability

- ✓ How can I prove that a computer can solve some specific problem?
  - E.g., sorting or the Traveling Salesperson Problem
- ✓ How can I prove that **no** computer can solve a specific problem?
  - If I prove it today, will it still be true tomorrow?

Describing the Problems to be Solved

Assumption 1: Inputs are *finite strings from some finite alphabet of characters.*

- ✓ `"37"`
- ✓ `"3*7=21"`
- ✓ `"sort([3,1,4,1,2], [1,2,3,4])."`

Assumption 2: *We care about decision problems* (i.e., answer is just yes or no)

- ✓ Is 37 prime?
- ✓ Does $3 \times 7 = 21$?
- ✓ Is $[1, 2, 3, 4]$ the result of sorting $[3, 1, 4, 1, 2]$?
- ✓ $\vdots$

Justifying the Assumptions

Assumption 1: Inputs are *finite strings from a finite alphabet of characters.*

- ✓ Generalizes "everything's just a string of bits"

Assumption 2: We care about *decision problems* (i.e., answer is just yes or no)

- ✓ We can solve other problems by asking enough questions.
- ✓ E.g., how could I figure out 3×7 by asking yes/no questions?

Combining the Assumptions

Any decision problem is equivalent to asking

"Is the input a member of the set L?"

for a suitable set L.

- ✓ For primality testing, $L = \{2, 3, 5, 7, 11, 13, 17, 19, \ldots\}$.
- ✓ For multiplication-correctness, $L = ?$

Conclusion

A one-to-one correspondance:

(computational) problems to solve

$\updownarrow$

sets of finite strings.

These sets are called “formal languages”

Formal Languages: Alphabets

An *alphabet* Σ is a finite, nonempty set.
The elements of Σ are called *letters* or *symbols*.

- ✓ Examples?

Formal Languages: Strings

A *string over* Σ is a sequence

$$c_1 c_2 \cdots c_n$$

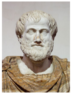
where $n \geqslant 0$ and each $c_i \in \Sigma$.

- ✓ Every string is *finite*, but could be arbitrarily long!
- ✓ We will write strings without quotation marks

  xyzzy

- ✓ We write the empty string as ε or λ

  Note: $\varepsilon, \lambda \notin \Sigma$!



What about my alphabet?

Languages

A "*language* L over Σ" is a set of strings over Σ.

- The set of all strings over Σ is written Σ^*.
- The empty set $\emptyset$ is a language.
- The non-empty set $\{\varepsilon\}$ is a language.
- Languages may be infinite, *even if they contain only finite strings!*
- Other examples?

A First Uncomputability Result

- ✓ Every computer program can be represented as a string in $\{0, 1\}^*$.
- ✓ There are as many string/programs as there are natural numbers
- ✓ There are as many languages as there are real numbers!

If you choose a language "at random," what is the probability it can be decided by some program?

Idealized Computers

We need a precise (mathematical) definition, abstracting away details that might change.

- ✓ Operating System
- ✓ Processor speed
- ✓ Memory capacity
- ✓ Power source (electricity, natural gas, dilithium, ...)
- ✓ Construction materials (silicon, graphene, legos, ...)
- ✓ Programming language (Java, Racket, Prolog, HMMM, ...)
- ✓ Architecture (single core, multicore, manycore, GPU, VLIW, ...)
- ✓ Data representation (ASCII, Unicode, binary, trinary, ...)

Today's Idealized Computer

State Machine, which

- ✓ A set of possible "configurations" (states)
- ✓ Rules for how the system proceeds from one state to another
  - Depends on current state *and* current input
- ✓ Accept or reject, based on the input this far.

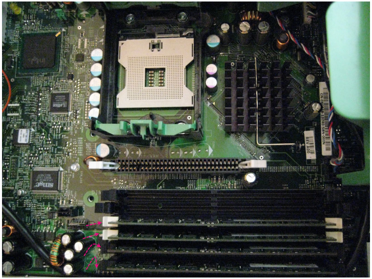
WHAT IS A STATE?

The *state* of a system at some instant consists of the all internal information needed to figure out how to proceed.

State?

http://www.flickr.com/photos/alexbrn/5035170693

State?



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Many systems have
more than 50 states!

Finite State Machines

A *finite state machine* is a state machine with finitely many possible states.

Finite State?

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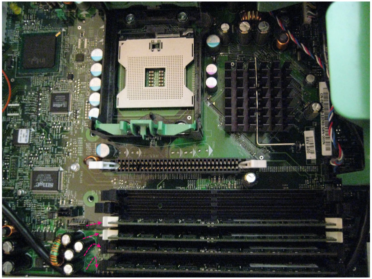
Finite State?



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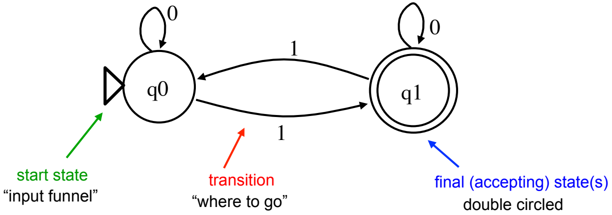
Deterministic Finite State Machines (DFAs)

Every state has one transition for *every* possible input symbol.

Often represented graphically:

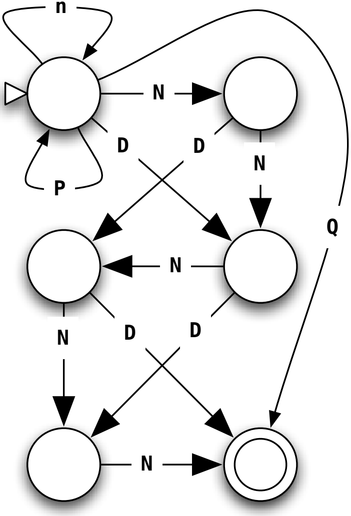


Does this machine accept 100? 1010?

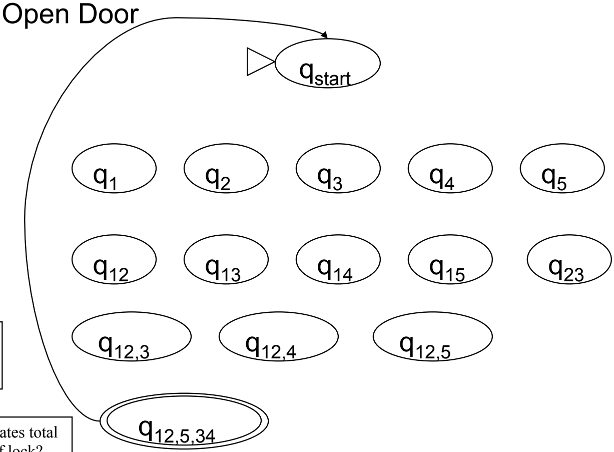
What is the language of this machine?

Finite states = finite memory. What are we "remembering" in each state?

DFAs are everywhere!



DFAs are everywhere!



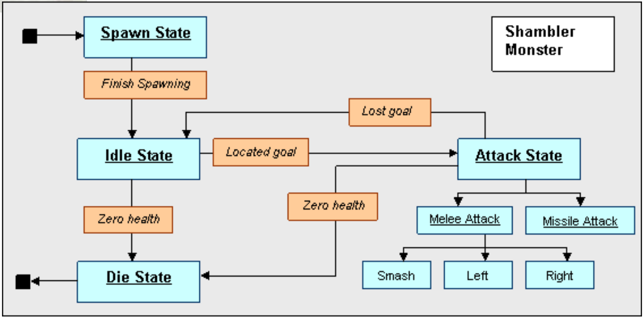
There's a lot missing here!

How many states total in this kind of lock?

DFAs are everywhere!

FSM == FearSoMe?

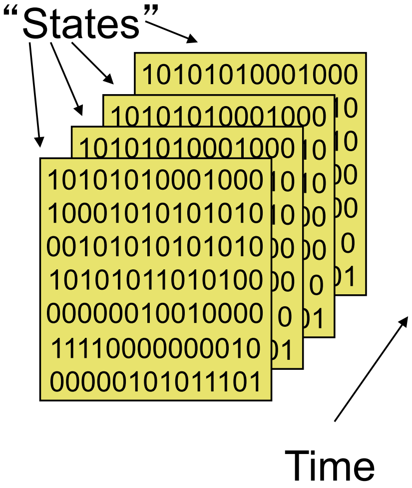
The FSM controlling Shambler monsters…



I'm *Quaking* in my AstroBoots

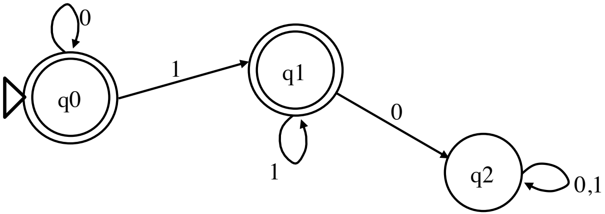
DFAs are everywhere!

Computers: State changes over time



Another DFA

What is the language of this machine?



Finite states = finite memory. What are we "remembering" in each state?

"Quiz" NAME:

What DFA *accepts* $L = \{ w \in \{0, 1\}^* \mid w$ is a binary number that's *even* $\}$

DFA for $L = \{ w \in \{0, 1\}^* \mid w$ *does not contain the substring* **101** $\}$?

More Examples

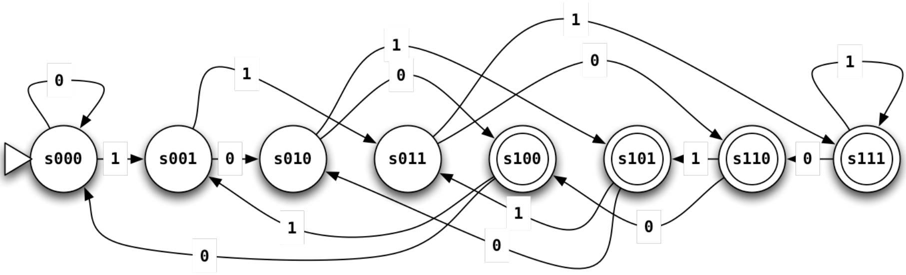
What DFA accepts

$$L = \{ w \in \{0, 1\}^* \mid w\text{'s third character is a } 1 \}$$

More Examples

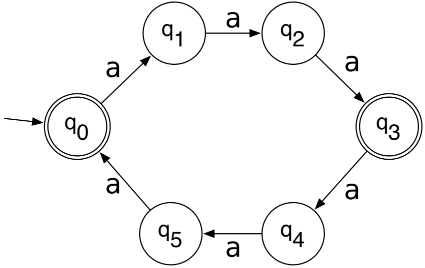
What DFA accepts

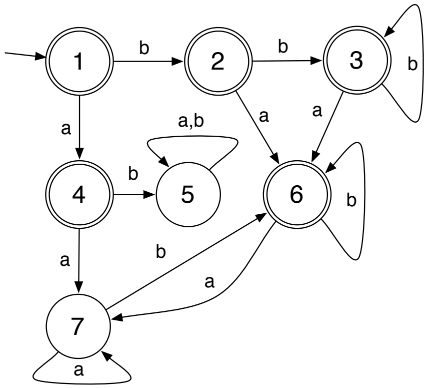
$$L = \{ w \in \{0,1\}^* \mid w\text{'s } \textbf{third-to-last} \text{ character is a } 1 \}$$



DFA SIZES

How can we know whether a DFA is as small as possible?





A DFA is minimal if *every* state is necessary (all states are distinguishable)

Distinguishing Strings

Recall

$$L = \{\, w \in \{0,1\}^* \mid w\text{'s third character is a } 1 \,\}$$

.

Meet our four string couples for today:

Couple #1	Couple #2	Couple #3	Couple #4
$w_1 = 011$	$w_1 = 011$	$w_1 = 01$	$w_1 = 0$
$w_2 = 111$	$w_2 = 110$	$w_2 = 10$	$w_2 = 10$

If we continue on with more digits, are the "fates" of these couples the *same* or *different?*

String Matchmaking, Formalized

Given a language $L \subset \Sigma^*$

Two strings w_1 and w_2 are **distinguishable** if

there exists $z \in \Sigma^*$ such that $w_1 z \in L$ and $w_2 z \notin L$ (or vice versa)

A set $S = \{w_1, \ldots, \ldots\}$ is **pairwise distinguishable** with respect to language L if

w_i and w_j are distinguishable for all $i \neq j$.

More Distinguishability Practice

$$L = \{\, 0^i 1^j \mid i \text{ is even}, j \text{ is odd} \,\}$$

Are the following distinguishable?

1. 00 and 0
2. 0000 and 00
3. 010 and 1
4. λ and 001

Distinguishability Theorem

Key insight: if w_1 and w_2 are distinguishable, they cannot both lead to the same state in our state machine.

Theorem

If there is a pairwise distinguishable set of N *states for a language* L, *then a DFA accepting* L *must have* $\geqslant N$ *states.*

Example

Prove that any DFA recognizing

$$L = \{ w \in \{0, 1\}^* \mid w\text{'s third character is a } 1 \}$$

must have at least 5 states.

Proof: $\{ \lambda, 1, 0, 11, 10 \}$ Proof: $\{ \lambda, 0, 00, 000, 001 \}$

Theorem

If there is a pairwise distinguishable set of N states for a language L, then a DFA accepting L must have $\geqslant N$ states.

Nonregular Language Theorem

A language is said to be **regular** if it can be recognized by some DFA.

Consider the language

$$L = \{\, 0^n 1^n \mid n \geqslant 0 \,\}$$

To show it's not regular, …

Nonregular Language Theorem

A language is said to be **regular** if it can be recognized by some DFA.

Consider the language of correct multiplications of natural numbers:

$$\begin{aligned} \mathrm{L} = \{ \quad & 0{\times}0{=}0,\ \ 0{\times}1{=}0,\ \ \ldots \\ & 1{\times}0{=}1,\ \ 1{\times}1{=}1,\ \ \ldots \\ & 2{\times}0{=}1,\ \ 2{\times}1{=}2,\ \ \ldots \\ \} \end{aligned}$$

(What is the alphabet?)

Prove that this language is not regular.

Conclusion

We can identify problems that no finite state machine can solve.

- ✓ $L = \{ 0^n 1^n \mid n \geqslant 0 \}$
- ✓ $L = \{ ww \mid w \in \{0,1\}^* \}$

If computers are finite state machines, then no computer can solve them either!

A HMC DFA (AUTHOR UNKNOWN)

Complete the finite-state machine below with at least three states and transitions of your own design.
Feel free to ignore or modify the sample states/transitions provided.