Worksheet: Cardinality

Cantor defined $|S| \leqslant |T|$ when there is an injective (one-to-one) mapping from S to T. A set S is said to be *countable* if $|S| \leqslant |\mathbb{N}|$, and *countably infinite* if $|S| = |\mathbb{N}|$ (i.e., $|S| \leqslant |\mathbb{N}|$ and $|\mathbb{N}| \leqslant |S|$). A set is *uncountable* if it is not countable.

What are some ways to show that an infinite set is countably infinite?

Which of the following sets are countable?

- The even natural numbers $(0, 2, 4, 6, \text{etc.})$

- All ordered pairs of natural numbers $(\langle 0, 17\rangle, \langle 99999, 62\rangle, \text{etc.})$

- All nonnegative rational numbers $(0, 3, \frac{13}{4}, \frac{16}{5}, \frac{22}{7}, \frac{355}{113}, \text{etc.})$

- The integers $(\ldots -3, -2, -1, 0, 1, 2, 3, \text{etc.})$

- All rational numbers
  $(0, -3, \frac{13}{4}, -\frac{16}{5}, \frac{22}{7}, -\frac{355}{113}, \text{etc.})$

- All finite sets of natural numbers
  $(\emptyset, \{3, 99, 127\}, \{2\}, \{1, 3, 5, 7, 9, 11, \ldots, 989877\}, \text{etc.})$

- All "finite strings" (finite sequences) of 0's and 1's
  (ϵ, 0, 10101011, 0000000, etc.)

- All "finite strings" (finite sequences) of a's and b's
  (ϵ, a, babababb, aaaaaaa, etc.)

- The dyadic rational numbers between 0 and 1 inclusive ($0, 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}$, etc.)

- All computer files

- All finite ordered sequences (i.e., lists) of natural numbers
  ($\langle\rangle, \langle 3, 3, 3, 99, 127, 3\rangle, \langle 2\rangle, \langle 2, 2\rangle, \langle 1, 3, 5, 7, 9, 11, \ldots, 989877, 1\rangle$, etc.)

- All polynomials with rational coefficients.

- All predicate logic formulas

- All Python programs

- All natural numbers that can be computed/printed by a Python program

- All real numbers between 0 and 1 that can be computed (printed?) by a Python program

- All co-finite sets (sets whose *complement* is finite) of natural numbers ($\{0, 1, 2, 3, \ldots\}, \{0, 1, 2, 4, 5, \ldots, 98, 100, \ldots, 126, 128, 129, \ldots\}, \{0, 1, 3, 4, 5, 6, \ldots\}$, etc.)

- All (countably) infinite sequences of 0's and 1's
((0,0,0,0,...), (0,1,0,1,0,1,...), (1, 0, 1, 0, 0, 1, 0, 0, 0, 1, ...), etc.)

- All sets of natural numbers

- All (countably) infinite sequences of digits
((0,0,0,0,...), (0,1,0,1,0,7,...), (7, 2, 1, 9, 9, 9, 9, 9, 9, 9, ...), etc.)

- The set of real numbers between 0 and 1 inclusive

- The set $\mathbb{R}$ of all real numbers

- The algebraic real numbers (i.e., zeros of polynomials with rational coefficients)