Assignment 3: Predicate Logic

Due: 11:00am, Wednesday, September 21

- Emails about this assignment should be directed to `cs81help@cs.hmc.edu`.
- The usual collaboration rules apply. You may *discuss* an exercise with any other student(s) currently taking CS 81 as long as:
  - You contribute equally;
  - You come away from this discussion only with *understanding in your head* — no written materials or computer notes may be retained;
  - Your submission is authored solely by you, on a separate occasion.
- You should refer only to materials from this semester of CS 81 (lecture notes, handouts, textbooks, grutors, profs, etc.).
- Bring a writeup/printout to class on the due date. Illegible answers will get no credit. (For this reason, the grutors like you to use LaTeX. But it's up to you.)
- Make sure your submission includes your name!

Reading

Review Sections 2.1–2.3 of Huth & Ryan. Pay particular attention to the sample proofs in Section 2.3; make sure you understand how they fit together.

1 Thinking in Predicate Logic

Do Exercise 2.1.4(a-k), found on page 158 of Huth and Ryan.

2 Formal Logic

Give natural deduction proofs of the following:

1. $\exists x.\,(R(x) \to B(x)) \vdash (\forall x.\,R(x)) \to (\exists x.\,B(x))$

   (e.g., $R(x)$ = "x is red", $B(x)$ = "x is bacon-flavored")

2. $(\forall x.\,L(x)) \lor (\forall x.F(x)) \vdash \forall x.\,(L(x) \lor F(x))$

   (e.g., $L(x)$ = "x is lost", $F(x)$ = "x is found")

3. $\forall x.\,(I(x) \lor U(x)) \vdash (\forall x.\,I(x)) \lor (\exists x.U(x))$

   (e.g., $I(x)$ = "x is interesting", $U(x)$ = "x is unremarkable")

4. $\forall x.\,(p \to Q(x)) \vdash p \to \forall x.Q(x)$

   (e.g., p = "I'm wearing earplugs", $Q(x)$ = "x is quiet")

5. $\vdash \exists x.\,(D(x) \to \forall y.\,D(y))$

   (The "drinker's paradox": there is at least one person satisfying "if he/she drinks beer, then everyone drinks beer.")

6. $\neg(\exists x.\,U(x)) \lor (\forall x.\,V(x)),\ p \to \forall x.\,D(x)\ \vdash\ \forall y.\,\forall z.\,((\neg U(z) \lor V(y)) \land (p \to D(y)))$

   (e.g., $U(x)$ = "x is a unicorn", $V(x)$ = "x is a vegetable", p = "I'm hungry", $D(x)$ = "x is delicious")

7. $\forall x.\,\neg S(x,x),\ \forall x.\,\forall y.\,\forall z.\,S(x,y) \land S(y,z) \to S(x,z)\ \vdash\ \forall x.\,\forall y.S(x,y) \to \neg S(y,x)$

   (e.g., $S(x,y)$ = "movie x is a sequel to movie y")

3 Informal Logic

Here is some background in set theory: for all sets X and Y and all elements z,

$$\begin{aligned}
&z \notin X \iff \neg(z \in X)\\
&z \in (X \cup Y) \iff (z \in X) \lor (z \in Y)\\
&z \in (X \cap Y) \iff (z \in X) \land (z \in Y)\\
&z \in (X \setminus Y) \iff (z \in X) \land \neg(z \in Y)\\
&X \subseteq Y \iff \forall x.\,(x \in X \to x \in Y)\\
&X = Y \iff (X \subseteq Y) \land (Y \subseteq X)\\
&X \text{ meets } Y \iff \exists z.\,z \in (X \cap Y)
\end{aligned}$$

For each of the following two propositions, give

(a) a proof in *mathematical English*. (For the first one, you only need to complete the proof provided.) Your proof should be 1–3 paragraphs of English text, as one might find in a (very precise) math book or textbook.

The only mathematical symbols in your proof should be set-theory symbols ($\in$, $\notin$, $\subseteq$, $\cup$, etc.). Do not include any formal logic symbols ($\forall$, $\exists$, $\wedge$, $\neg$, etc.)

Your proof should be complete and without holes. They will probably not be great literature but, where possible, strive for clarity.

(b) Identify four natural deduction rules that correspond to (explicit or implicit) steps in your proof, and at least one part of each proof to which that rule corresponds.

1. $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$.

   (a) *Proof.* Let $x \in A \setminus (B \setminus C)$ be given. Then $x \in A$ and $x \notin B \setminus C$. That is, $x \notin B$ or $x \in C$. If $x \notin B$, then $x \in A \setminus B$; if $x \in C$ then $x \in A \cap C$. In either case, $x \in (A \setminus B) \cup (A \cap C)$. Thus $A \setminus (B \setminus C) \subseteq (A \setminus B) \cup (A \cap C)$.
   ... □

2. If A meets B, then $A \cup B \not\subseteq (A \setminus B) \cup (B \setminus A)$.

Extra Credit [10%]

The formula $\neg\neg p \to p$ is a tautology, but it is not provable in constructive (a.k.a. intuitionistic) logic. That is, to prove it you need to use a non-constructive rule like proof-by-contradiction, or the law of the excluded middle, or (for a trivial proof) $\neg\neg$-elimination.

On the other hand, even constructive logicians will agree that the formula isn't false; $\neg\neg(\neg\neg p \to p)$ *is* provable constructively, without any non-constructive rules. Provide a constructive proof. (You will need $\neg$-introduction, but that is a rule *different* from proof-by-contradiction.)

[This problem is nontrivial; don't spend too much time on it unless you find it interesting!]