Assignment 4: More Predicate Logic

Due: 11:00am, Wednesday, September 28

- Emails about this assignment should be directed to `cs81help@cs.hmc.edu`.
- The usual collaboration rules apply. You may *discuss* an exercise with any other student(s) currently taking CS 81 as long as:
  - You contribute equally;
  - You come away from this discussion only with *understanding in your head* — no written materials or computer notes may be retained;
  - Your submission is authored solely by you, on a separate occasion.
- You should refer only to materials from this semester of CS 81 (lecture notes, handouts, textbooks, grutors, profs, etc.).
- Bring a writeup/printout to class on the due date. Illegible answers will get no credit.
- Make sure your submission includes your name!

Review Section 2.4 of Huth & Ryan.

1 Proof or No Proof?

For each of the following, give a natural deduction proof if one exists. If there is no proof, you should

1. Define a model that makes the assumptions true and the conclusion false;
2. Briefly explain why the assumptions are true and the conclusions are false;
3. *Also*, find an alternate model that makes the conclusion *true*. (It doesn't matter whether this new model makes the assumptions true or not; this is just more practice with models.)

1. Assumption: $\forall x. \forall y. (S(x, y) \rightarrow S(y, x))$
   Conclusion: $\forall x. \neg S(x, x)$
2. Assumption: $\top$ (i.e., no assumption)
   Conclusion: $\forall x. \forall y. \big(S(x, y) \rightarrow \exists w. (S(x, w) \wedge S(w, y))\big)$
3. Assumption: $\forall x. \big((P(x) \rightarrow Q(x)) \wedge (Q(x) \rightarrow P(x))\big)$
   Conclusion: $(\forall x. P(x)) \rightarrow (\forall x. Q(x))$
4. Assumption: $(\forall x. P(x)) \rightarrow (\forall x. Q(x))$
   Conclusion: $\forall x. \big((P(x) \rightarrow Q(x)) \wedge (Q(x) \rightarrow P(x))\big)$
5. Assumption: $(\forall x. P(x)) \rightarrow q$
   Conclusion: $\forall x. (P(x) \rightarrow q)$
6. Assumptions: $\exists x. P(x), \qquad \exists y. Q(y)$
   Conclusion: $\exists z. (P(z) \wedge Q(z))$
7. Assumption: $(\exists x. P(x)) \vee (\exists y. Q(y))$
   Conclusion: $\exists z. (P(z) \vee Q(z))$
8. Assumption: $\forall x. \exists y. S(x, y)$
   Conclusion: $\exists y. \forall x. S(x, y)$

2 Bounded Quantifiers

In class, we mentioned that the "bounded quantifier" notation (where we restrict the individuals being quantified over) can be translate into the more primitive notions on the right.

$$
\begin{array}{lcl}
\exists x \in S.\, P(x) & \longrightarrow & \exists x.\, (x \in S \land P(x)) \\
\forall x \in S.\, P(x) & \longrightarrow & \forall x.\, (x \in S \to P(x)) \\
\\
\exists x \leqslant n.\, P(x) & \longrightarrow & \exists x.\, (x \leqslant n \land P(x)) \\
\forall x \leqslant n.\, P(x) & \longrightarrow & \forall x.\, (x \leqslant n \to P(x))
\end{array}
$$

It might be unexpected that the translations are different for the different quantifiers: bounded-$\exists$ becomes a logical-and, while the translation of bounded-$\forall$ becomes an implication.

1. Describe a model $\mathcal{M}_1$ (where the set of individuals is the set of $\mathbb{N}$ of natural numbers, and the relation $\leqslant$ is interpreted as the actual less-than-or-equal-to relation on $\mathbb{N}$) that makes

   $$\forall x \leqslant n.\, f(x) \leqslant m$$

   and hence

   $$\forall x.\, (x \leqslant n \to f(x) \leqslant m)$$

   true but makes

   $$\forall x.\, (x \leqslant n \land f(x) \leqslant m)$$

   false. (That is, complete the model by giving interpretations of the function f and the constants n and m.)

2. Describe a model $\mathcal{M}_2$ (where the set of individuals is $\mathbb{N}$ and the relation $\leqslant$ is interpreted as less-than-or-equal-to relation on $\mathbb{N}$) that makes

   $$\exists x.\, (x \leqslant n \to f(x) \leqslant m)$$

   true but makes

   $$\exists x \leqslant n.\, f(x) \leqslant m$$

   and hence

   $$\exists x.\, (x \leqslant n \land f(x) \leqslant m)$$

   false.

3 Mathematical English, Revisited

Read the attached short excerpt from Crossley's *Essential Topology*.[1]

1. Proposition 2.4, if made more explicit, would be something like "for all collections, if the collection consists of of open sets, then the union of that collection is open" (where "is open" has its own logical definition, in the box at the top of the page). Consider the proof.

   (a) If we translated this mathematical english into a formal natural deduction proof, what rule(s) correspond to the first sentence?

   (b) What rule(s) corresponds to the next phrase "if x lies in the union $x \in \bigcup_{i \in I} S_i$, ..."?

   (c) The proof implicitly uses extential elimination, twice. What are the two existentials, and what is the conclusion ("C") of the subproofs?

   [Hint: the second sentence is (in part) reminding you that by definition of the $\bigcup$ operator, the assumption $x \in \bigcup_{i \in I} S_i$ means $\exists i{\in}I.\ x{\in}S_i$. Also, the existential eliminations are nested so there's really one common conclusion.]

2. Identify four steps (explicit or implicit) in the proof of Lemma 2.8 that correspond to natural-deduction rules. (Each should be a different natural-deduction rule.)

3. Identify four steps (explicit or implicit) in the proof of Theorem 2.9 that correspond to natural-deduction rules. (Each should be a different natural-deduction rule.)

[1] Because the Claremont Library has a subscription, this book (like many other CS and Math texts published by Springer Verlag) is available as free PDFs at `http://springerlink.com/` (for personal use only). There's one PDF per chapter, but if you have the `pdftk` program installed, there's a very nice download script available at `http://milianw.github.com/springer_download/` to get books as single PDFs.