Assignment 5: Beyond Natural Deduction

Due: 11:00am, Wednesday, October 5

- Emails about this assignment should be directed to `cs81help@cs.hmc.edu`.
- The usual collaboration rules apply. You may *discuss* an exercise with any other student(s) currently taking CS 81 as long as:
  - You contribute equally;
  - You come away from this discussion only with *understanding in your head* — no written materials or computer notes may be retained;
  - Your submission is authored solely by you, on a separate occasion.
- You should refer only to materials from this semester of CS 81 (lecture notes, handouts, textbooks, grutors, profs, etc.).
- Bring a writeup/printout to class on the due date. Illegible answers will get no credit.
- Make sure your submission includes your name!

1 Tableau Proofs

For each of the following, give a tableau proof.

1. $\vdash \quad (p \to q) \lor (q \to r)$
2. $(p \land q) \to r,\ r \to s, q \land \neg s \quad \vdash \quad \neg p$
3. $(\exists x.\, P(x)) \to (\exists y. Q(y)) \quad \vdash \quad \exists z.\, (P(z) \to Q(z))$
4. $\forall x.\, \big((P(x) \to Q(x)) \land (Q(x) \to P(x))\big) \quad \vdash \quad (\forall x.\, P(x)) \to (\forall x.\, Q(x))$

2 Sequent Calculus

1. For each of the following, give a sequent-calculus proof. If you wish, you can treat the sequences of formulas as sets, and leave uses of the structural (exchange, weakening, and contraction) rules implicit except where they help for clarity. (For example, you might treat $p, q, r \vdash p$ as an axiom, rather than taking $p \vdash p$ and weakening twice on the left.)

   (a) $\vdash p \vee \neg p$

   (b) $\neg(p \wedge q) \vdash \neg p \vee \neg q$

   (c) $\vdash ((p \to q) \to p) \to p$

   (d) $\bot \vdash A$

2. Gentzen developed the sequent calculus as a method of proving results about natural deduction. He proved the following key facts (by induction and proof trees!):

   - For every natural deduction proof, there is a corresponding sequent-calculus proof with the same conclusion (possibly using the Cut rule) and vice-versa. Hence, both systems have the same theorems.
   - For every sequent-calculus proof using the Cut rule, there is a (possibly much longer) sequent-calculus proof of the same result that does not use the Cut rule. (This property of sequent calculus is called "Cut Elimination.")

   A logical system is said to be *consistent* if it is impossible to prove $\bot$ from no assumptions. Relying on Cut Elimination and the fact that sequent calculus proofs are (inductively generated) trees, prove that sequent calculus (and hence natural deduction!) is consistent.[1] That is, prove that $\vdash \bot$ is not a theorem of sequent calculus.

3 Resolution

Convert each of the following to clausal form, and show the resolution proofs:

1. $\vdash \quad (p \to q) \vee (q \to r)$

2. $(p \wedge q) \to r,\ r \to s, q \wedge \neg s \quad \vdash \quad \neg p$

[1] There's a completely different proof of consistency that relies on via soundness and completeness of natural deduction. The goal here is a "purely syntactic" proof of consistency that doesn't involve models at all, just proof trees.