Introduction to Computability

October 24, 2011

Let what you say be simply 'Yes' or 'No'; anything more than this comes from evil.

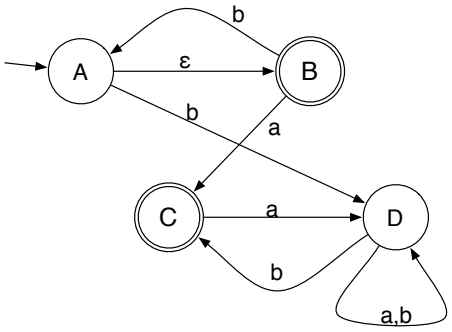
— Matthew 5:37 (English Standard Version)

It from bit. Otherwise put, every it . . . derives its function, its meaning, its very existence entirely. . . [from] answers to yes or no questions, binary choices, bits.

— John Archibald Wheeler, "It from Bit," *At Home in the Universe*, 1994.

Selected Topics in Computability

Finite State Machines



Regular Expressions

```
a(a|b)*a
```

Context-Free Grammars

```
E → n
E → E + n
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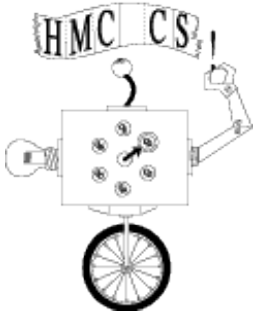
Turing Machines



Partial Recursive Functions

$h(\mu x.[f(g(x))=0])$

Lambda Calculus & Combinatory Logic (CS 131)

$\lambda f.\lambda b.f(f(b)$

$S(SII)(SII)KS$

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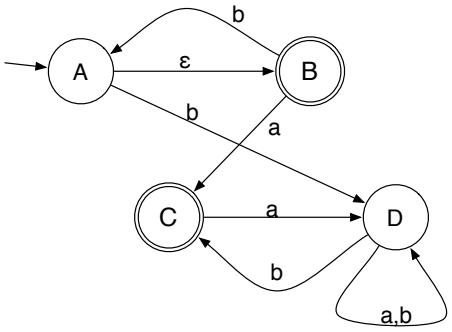

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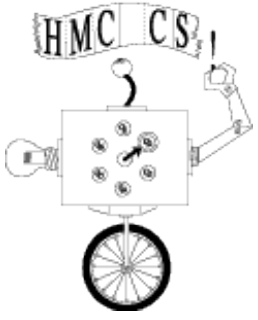

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If these are the answers, what is the question?

Common Simplifying Assumptions

1. Encoding inputs as *finite strings*.
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1. Encoding inputs as *finite strings*.
   - Generalizing "everything's just bits"
2. A focus on *decision problems*
   - Described as the set of inputs (strings) where the answer is "yes"

Alphabet

An *alphabet* Σ is a nonempty set.
The elements of Σ are called *letters* or *symbols*.

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The elements of Σ are called *letters* or *symbols*.

- ✓ We will always assume a finite alphabet.
- ✓ Examples?

Strings

A *string over* Σ is a sequence

$$c_1 c_2 \cdots c_n$$

where $n \geqslant 0$ and each $c_i \in \Sigma$.

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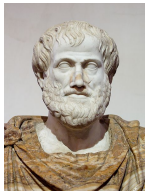

What about my alphabet?

Languages

A *language* L over Σ is a set of strings over Σ.

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- ✓ The set of all strings over Σ is written Σ^*.
- ✓ The empty set $\emptyset$ is a language.
- ✓ Languages may be finite or infinite, but they contain only finite strings!
- ✓ Other examples?

Strings (more formally)

The set Σ^* can be defined inductively:

- ✓ $\varepsilon \in \Sigma^*$
- ✓ If $a \in \Sigma$ and $x \in \Sigma^*$ then $a \bullet x \in \Sigma^*$

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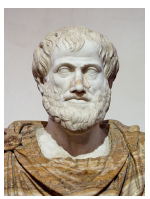
✓ 

Am I in Σ^* ?

Resulting Induction Principles

- ✓ **Structural induction on strings:** If $P(\varepsilon)$ and

$$\forall x \in \Sigma^*.\ \forall a \in \Sigma. \quad P(x) \to P(ax)$$

then $\forall w \in \Sigma^*.\ P(w)$.

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- ✓ **Induction by length:** If $P(\varepsilon)$ and

$$\begin{aligned} \forall n > 0. \quad & (\forall w \in \Sigma^*.\texttt{length}(w) = n{-}1 \to P(w)) \\ & \to (\forall w \in \Sigma^*.\texttt{length}(w) = n \to P(w)) \end{aligned}$$

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Resulting Induction Principles

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- ✓ **Strong induction by length:** If

$$\forall n \geqslant 0. \quad (\forall w \in \Sigma^*.\ \texttt{length}(w) < n \to P(w)) \\ \to (\forall w \in \Sigma^*.\ \texttt{length}(w) = n \to P(w))$$

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- ✓ **Structural induction on strings:** If $P(\varepsilon)$ and

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The inductive definition of strings also justifies defining *functions* by induction/recursion.

Operations on Strings

```
append(ε,y)  := y
append(ax,y) := a•append(x,y)

length(ε)  := 0
length(ax) := 1 + length(x)

rev(ε)  := ε
rev(ax) := append(rev(x),a)
```

Operations on Strings

```
append(ε,y)  := y                    εy     := y
append(ax,y) := a•append(x,y)        (ax)y  := a(xy)

length(ε)  := 0                      |ε|  := 0
length(ax) := 1 + length(x)          |ax| := 1 + |x|

rev(ε)  := ε                         ε^R    := ε
rev(ax) := append(rev(x),a)          (ax)^R := x^R a
```

Operations on Strings

`append(ε,y) := y`	$\varepsilon y := y$
`append(ax,y) := a•append(x,y)`	$(ax)y := a(xy)$
`length(ε) := 0`	$\lvert\varepsilon\rvert := 0$
`length(ax) := 1 + length(x)`	$\lvert ax\rvert := 1 + \lvert x\rvert$
`rev(ε) := ε`	$\varepsilon^R := \varepsilon$
~~`rev(ax) := append(rev(x),a)`~~	~~$(ax)^R := x^R a$~~

Prove

- ✓ $\forall w \in \Sigma^*.\ (w\varepsilon = w)$
- ✓ $\forall w \in \Sigma^*.\forall y, z \in \Sigma^*.\ (wy)z = w(yz)$
- ✓ $\forall w \in \Sigma^*.\forall z \in \Sigma^*.\ \texttt{length}(wz) = \texttt{length}(w) + \texttt{length}(z)$
- ✓ $\forall w \in \Sigma^*.\forall z \in \Sigma^*.\ (wz)^R = z^R w^R$

Operations on Languages

$$L \cup M$$

$$L \cap M$$

$$L \setminus M$$

$$LM := \{ xy \mid x \in L,\ y \in M \}$$

$$L^n \qquad L^0 := \{\varepsilon\}$$
$$L^{n+1} := LL^n$$

$$L^* := L^0 \cup L^1 \cup L^2 \cup \cdots$$

$$L^+ := L^1 \cup L^2 \cup \cdots$$

Example

Assume $L = \{ \texttt{ba}, \texttt{da} \}$ and $M = \{ \texttt{da}, \texttt{rk}, \varepsilon \}$.

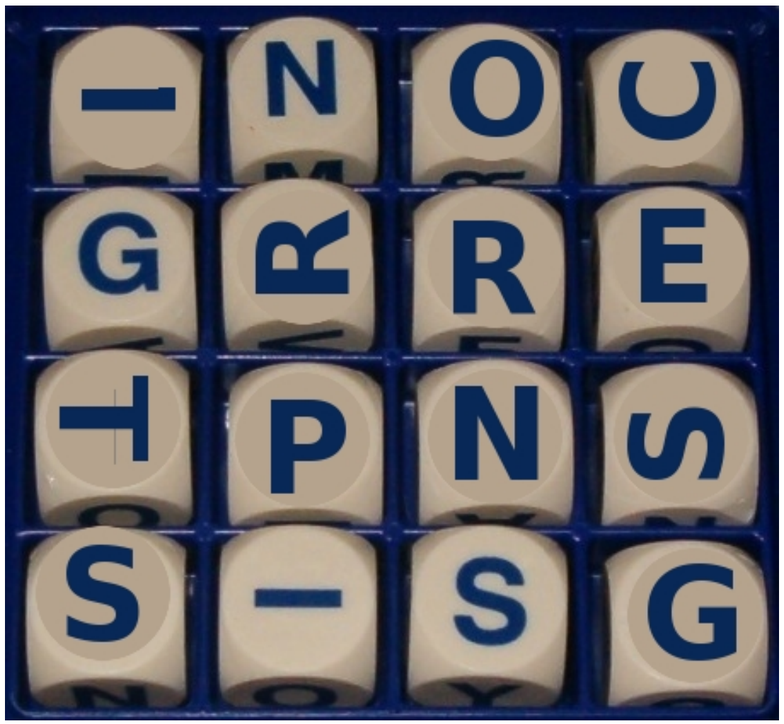
$$
\begin{aligned}
L \cup M &= \\
L \cap M &= \\
L \setminus M &= \\
LM &= \\
L^3 &= \\
L^* &= \\
L^+ &=
\end{aligned}
$$

Language Equivalences (not the same L)

- ✓ $L\emptyset =$
- ✓ $L\{\varepsilon\} =$
- ✓ $\{\varepsilon\}^* =$
- ✓ $\{\varepsilon\}^+ =$
- ✓ $\emptyset^* =$
- ✓ $\emptyset^+ =$
- ✓ $(L \cup M)N =$
- ✓ $(L^*)^* =$

How To Cheat at Boggle

Tracey (Boggle cheater)



Each word must:

be longer than three letters

be formed from adjacent tiles

use each tile at most once