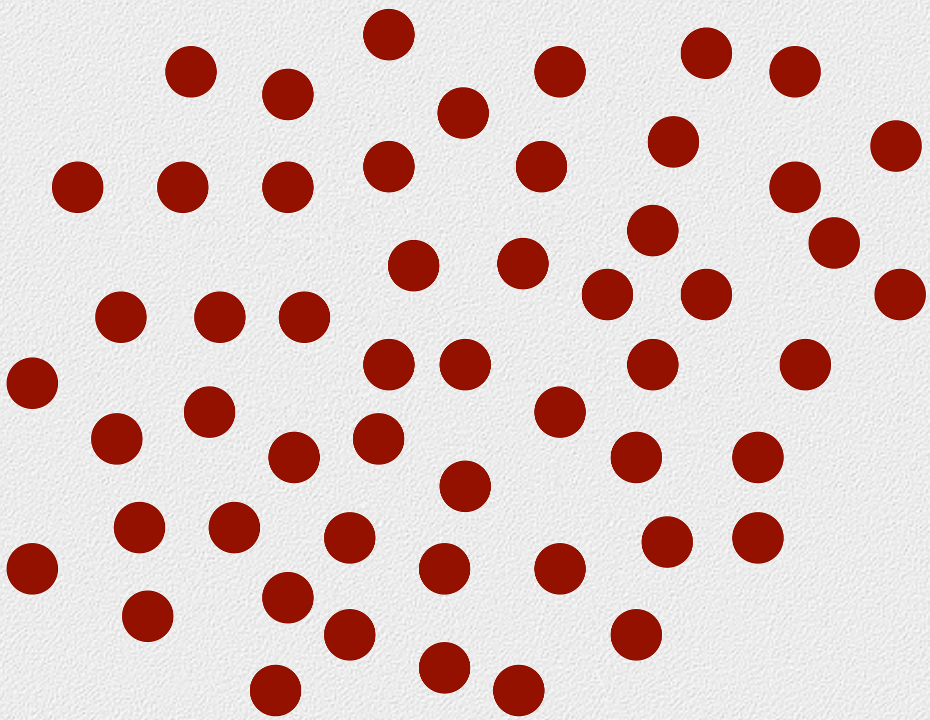
Proofs: Natural Deduction

September 5, 2011

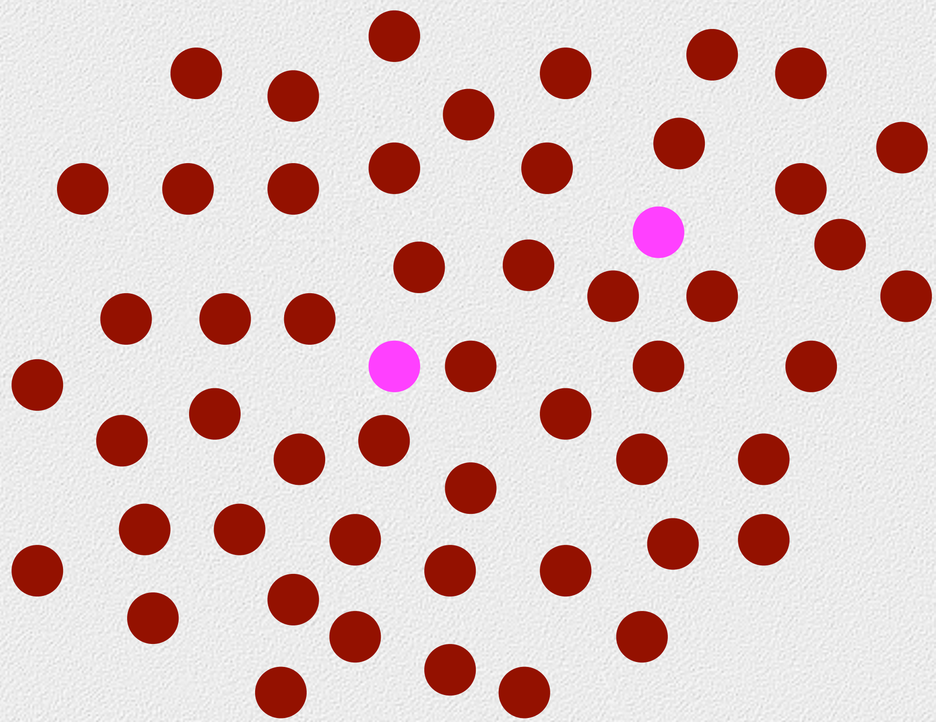
CS 81: Computability and Logic

Review

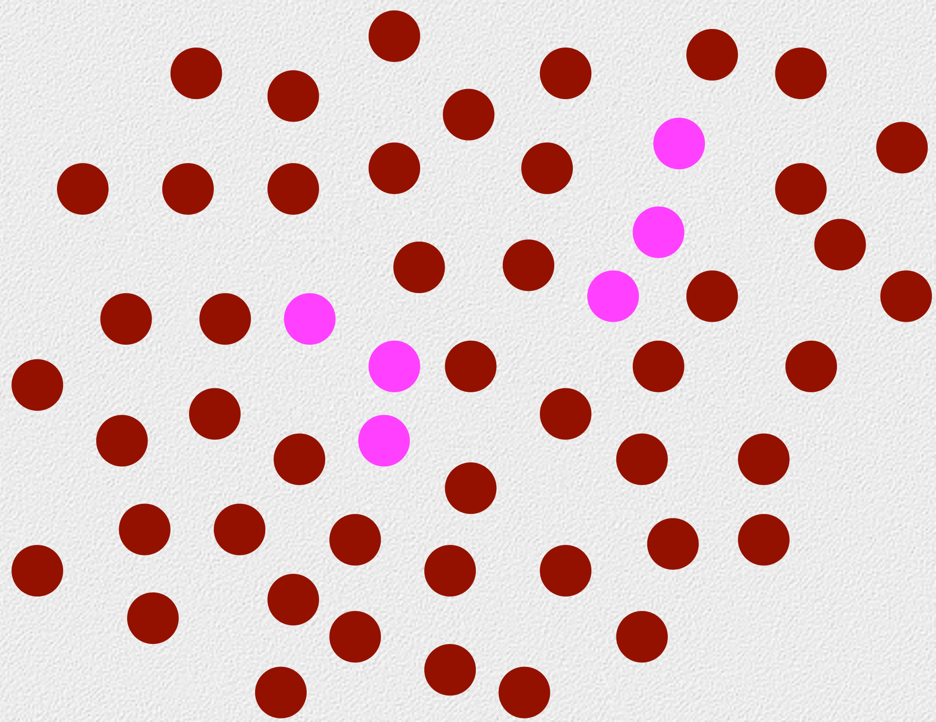
Inductively Defined Sets



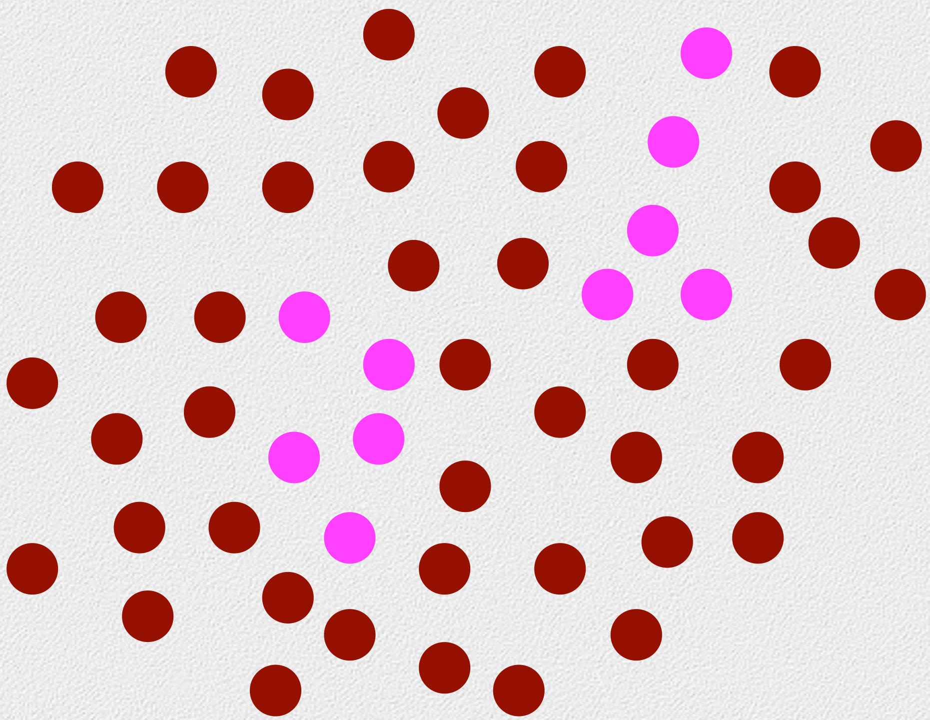
Base Case



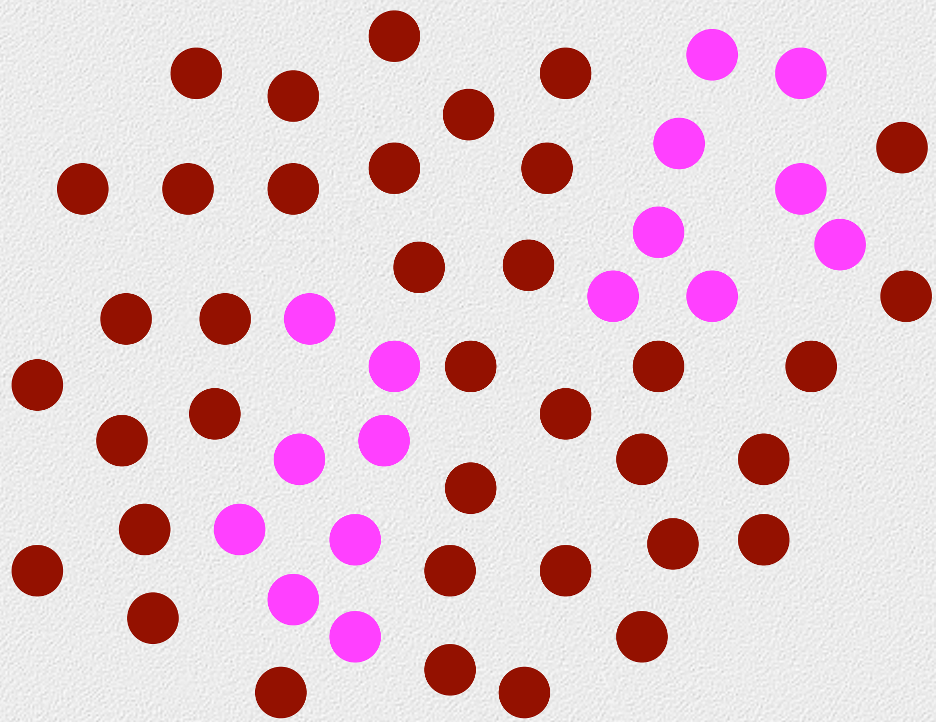
Applying Rules: 1x



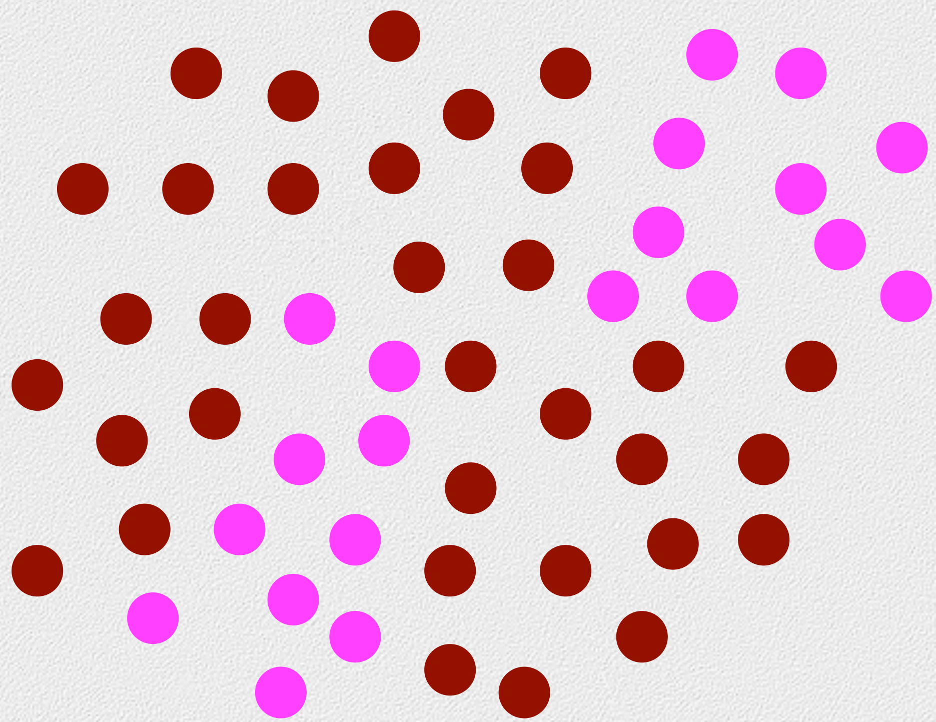
Applying Rules: 2x



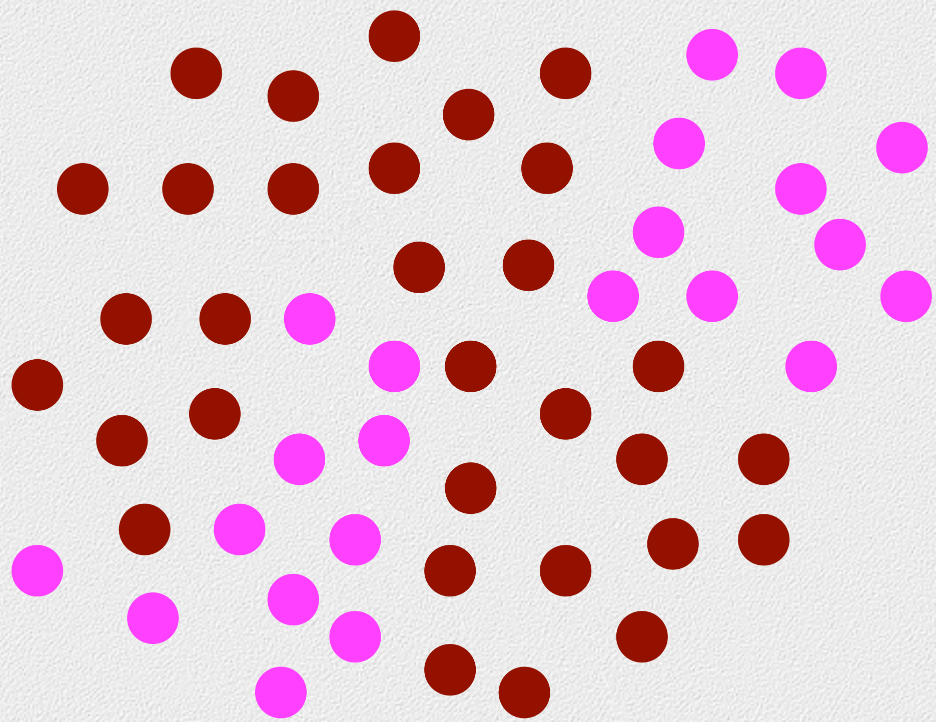
Applying Rules: 3x



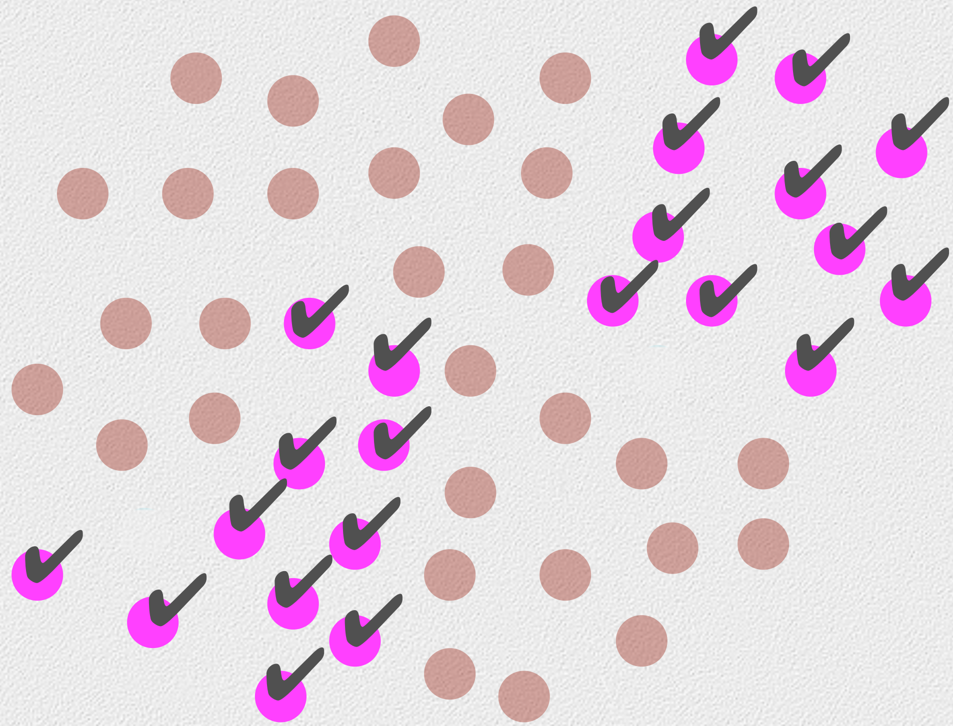
Applying Rules: 4x



Applying Rules: in the limit



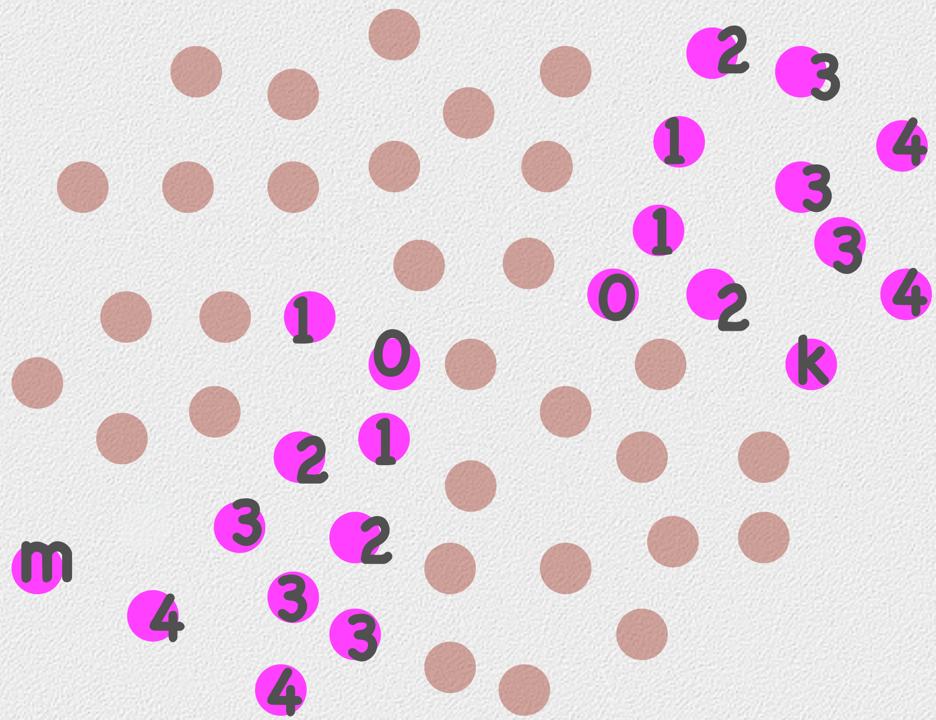
Structural Induction



Base case(s) have the property;

Rules preserve the property.

Numeric Interpretation



The stage 0 items have the property;

If items introduced at stage n (or $\leq n$) have the property, so do the items at stage $n+1$.

Well-Formed Formulas

(an inductive definition!)

p, q, r, etc., are WFFs.

$\top$ and $\bot$ are WFFs.

If A is a WFF then so is $\neg$A.

If A and B are WFFs then so is (A$\wedge$B).

If A and B are WFFs then so is (A$\vee$B).

If A and B are WFFs then so is (A$\rightarrow$B).

Simplifications

We will sometimes omit parentheses:

Binding tightest to loosest: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$

Left-associative: $\wedge$, $\vee$, $\leftrightarrow$

Right-associative: $\rightarrow$

$\neg p \wedge q \rightarrow r$

$p \rightarrow q \rightarrow p$

$\neg p \wedge r \vee p \wedge \neg q$

$(p \rightarrow q \rightarrow r) \rightarrow (p \rightarrow q) \rightarrow p \rightarrow r$

Exercise

Choose specific propositions for p and q, and express these formulas in English.

$p \wedge q \rightarrow q \wedge p$

$p \rightarrow p$

$p \vee \neg p$

$\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q$

$\neg\neg(p \vee \neg p)$

Theorems

(An inductively defined set!)

Axioms/Axiom Schemes:

... is a theorem.

... is a theorem.

Rules of Inference:

If ... is a theorem and ... is a theorem then ... is also a theorem.

Truth vs. Provability

A is true if B & C &D are:

$$B,C,D \models A$$

A is provable if B & C & D are:

$$B,C,D \vdash A$$

How should these relate?

Example:

Axiom Schemes (For any formulas F, G, and H):

1. $\vdash F \rightarrow (G \rightarrow F)$
2. $\vdash (F \rightarrow G \rightarrow H) \rightarrow (F \rightarrow G) \rightarrow F \rightarrow H$
3. $\vdash (\neg G \rightarrow \neg F) \rightarrow (F \rightarrow G)$

Rule of Inference: (For any formulas F and G):

If $\vdash F$ and $\vdash F \rightarrow G$ then $\vdash G$.

Theorems

(An inductively defined set!)

Axioms/Axiom Schemes:

... is a theorem.

... is a theorem.

} true?

Rules of Inference:

If ... is a theorem and ... is a theorem
then ... is also a theorem.

} truth-preserving?

Natural Deduction

- A way of organizing rules of inference that approximates the reasoning of “real” mathematical proofs.
- General approach: for each logical symbol, “introduction” and “elimination” rules.