Reductions and Undecidability

November 21, 2011
CS 81: Computability & Logic

An engineer and a mathematician are out for a walk when they spot a house on fire. There is a hose lying nearby; the engineer quickly attaches it to a fire hydrant and puts out the fire. They continue walking. The mathematician spots a house where there is a hose attached to a hydrant. He detaches the hose and sets the house on fire, thus reducing it to a previously solved problem.

Recall: TMs and Languages

- ✓ A TM **accepts** a string if that input leads to the "**y**" state.

- ✓ A language is **semidecidable** (a.k.a. semidecidable, recursively enumerable) if there is a TM that accepts exactly the strings in the language.

- ✓ A language is **decidable** (a.k.a. recursive) if it is accepted by a TM that always halts (i.e., the TM always ends up in **y** or **n**).

Recall: Decidable vs. Semidecidable

- ✓ If a language is decidable, then its complement is decidable. Why?
- ✓ If a language is semidecidable, and its complement is semidecidable, then the language is decidable. Why?

Turing Machines as Enumerators

Several variant definitions. Each specify a language L.

1. A TM that prints out all the members of L, *one at a time* (but not necessarily in any particular order)
2. A TM that prints out all the members of L, *one at at time* (but...) with arbitrarily many repeats.
3. A TM that, *given an integer* n, returns the nth *element of a sequence* like (1) *above*.
4. A TM that, *given an integer* n, returns the nth *element of a sequence* like (2) *above*.

For a fixed language, all these are interconvertable.

Theorem

A language is semidecidable $\iff$ *it can be enumerated.*

A language is decidable $\iff$ *it can be enumerated in lexicographic order.*

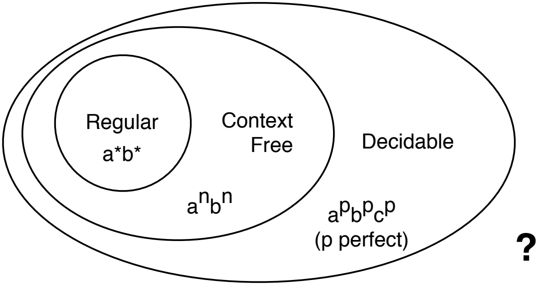
Church-Turing Thesis

If it can be done at all, then it can be done by

- ✓ A Turing Machine
- ✓ Lambda Calculus
- ✓ An Unrestricted Grammar
- ✓ A 2-register machine
- ✓ C
- ✓ ...

(Note: assumes suitably coded inputs and outputs)

Is There More?



Languages of Acceptance

Which are semidecidable (by a TM)? Decidable?

- ✓ $A_{\text{DFA}} = \{ \langle D, w \rangle \mid D \text{ a DFA}, D \text{ accepts } w \}$
- ✓ $A_{\text{NFA}} = \{ \langle N, w \rangle \mid N \text{ an NFA}, N \text{ accepts } w \}$
- ✓ $A_{\text{RE}} = \{ \langle R, w \rangle \mid R \text{ a regexp}, R \text{ matches } w \}$
- ✓ $A_{\text{CFG}} = \{ \langle G, w \rangle \mid G \text{ a CFG}, G \text{ produces } w \}$
- ✓ $A_{\text{TM}} = \{ \langle M, w \rangle \mid M \text{ a TM}, M \text{ accepts } w \}$

Digression: Bootstrapping a Compiler

Lots of compilers are written in the same language they compile!

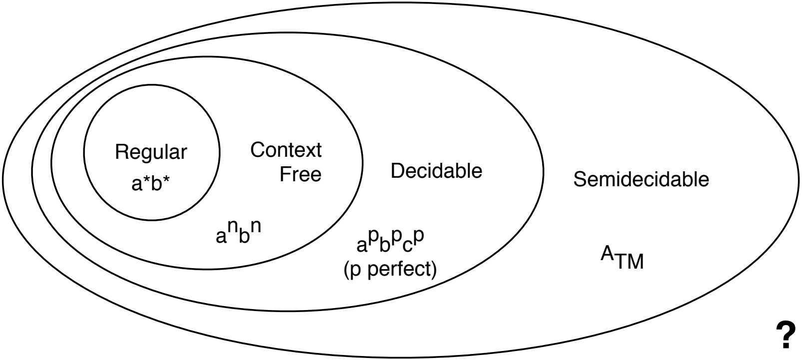
- ✓ Gnu C Compiler (used in CS 105) is written in C
- ✓ Glasgow Haskell Compiler (used in CS 131) is in Haskell
- ✓ *etc.*

Practical reasons to run programs on their own source code!

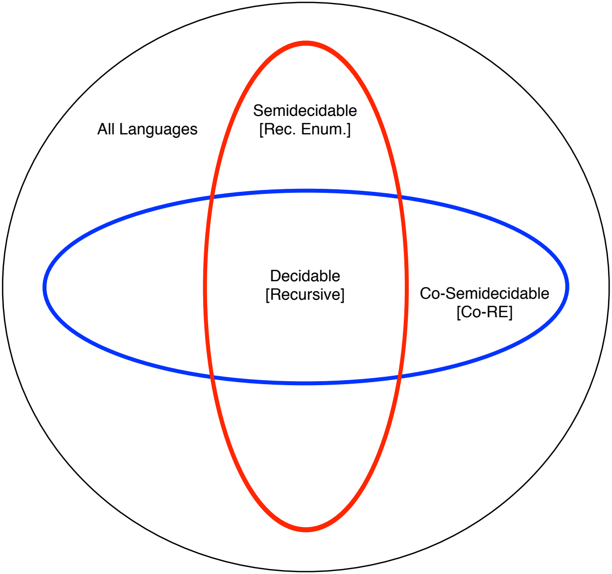
A_{TM} IS NOT DECIDABLE

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$
M_0	Acc.		Acc.	Acc.		
M_1						
M_2	Acc.			Acc.		
M_3	Acc.	Acc.	Acc.	Acc.	Acc.	
M_4			Acc.	Acc.		

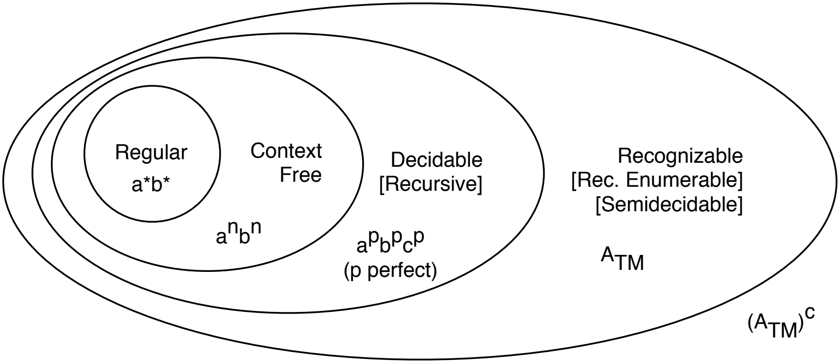
Is There More?



What is the complement of A_{TM}?



We'll Stop Here



Obligatory Corollary

Theorem

The language

$$H = \{ \langle M, w \rangle \mid M \text{ a TM}, M \text{ halts on } w \}$$

is not decidable.

Proof.

Suppose there were a halt-checking TM… $\square$

Undecidability, So Far

1. Acceptance for TMs is semidecidable but not decidable.

$$A_{\mathrm{TM}} = \{\, \langle M, w \rangle \mid M \text{ a TM}, M \text{ accepts } w \,\}$$

2. *Non*-Acceptance for TMs is not semidecidable (hence not decidable).

3. TM Halting is semidecidable but not decidable.

$$H = \{\, \langle M, w \rangle \mid M \text{ a TM}, M \text{ halts on } w \,\}$$

Reductions

Given problems P and Q, we say that

$$\mathrm{P} \leqslant \mathrm{Q}$$

if a solution to Q would let us solve P as well.

- ✓ I.e., (P is "not fundamentally more difficult" Q.)
- ✓ We say "P **reduces to** Q."

In Math, *we often show we can solve a problem* X by taking advantage of previously-solved problem Q (i.e., *prove* $\mathrm{X} \leqslant \mathrm{Q}$)

In Theocomp, we typically prove a problem X *hard* by showing a solution to X would also solve the hard problem P (i.e., prove $\mathrm{P} \leqslant \mathrm{X}$).

Reductions

To prove that P reduces to Q ($P \leqslant Q$), it suffices to prove:

- ✓ If we have a solver for Q, then we can use it to solve any instance of P.
- ✓ I.e., show you could construct a P-solver if you could make calls to a Q-solving subroutine.

Commonly, we instead prove a "mapping reduction":

- ✓ For every instance of P, we can contruct an instance of Q with the same yes/no answer.

Why is this enough?

Warning

It is *easy* to get the reduction backwards!

These are *essentially* proof by contradiction:

- ✓ Assume the unknown problem X is *decidable/semidecidable*
- ✓ Show that it means that H (or A_{TM} or ...) is *decidable/semidecidable*.
- ✓ Contradiction.
- ✓ Therefore, X is not *decidable/semidecidable*.

If you *ever* find yourself *assuming* things known to be false ("assume I had a halt checker...then I could solve X") you're doing the reduction wrong.

Reduction Practice

Show the following are not decidable (e.g., by reducing A_{TM} **to** each).

- ✓ $NE_{TM} := \{ \langle M \rangle \mid M$ accepts at least one $w \in \Sigma^* \}$
- ✓ $E_{TM} := \{ \langle M \rangle \mid L(M) = \emptyset \}$
- ✓ $ALL_{TM} := \{ \langle M \rangle \mid L(M) = \Sigma^* \}$
- ✓ Accepts-s$:= \{ \langle M \rangle \mid M$ accepts $s \}$
- ✓ Regular $:= \{ \langle M \rangle \mid L(M)$ is regular $\}$

Functional vs. Structural Properties

Structural questions (often, but not always, decidable):

- ✓ Does this TM have more than 500 control states?
- ✓ Does this TM ever use more than 300 cells of tape for some input?
- ✓ Does this TM ever write the symbol a ?

Functional questions (rarely decidable):

- ✓ Does this TM accept the empty language?
- ✓ Does this TM accept a finite language?
- ✓ Do all the strings accepted by this TM have even length?
- ✓ In general:

Given a TM T, does $L(T)$ have property P?
Given a semidecidable L, does L have property P?

Rice's Theorem

No *functional* and *nontrivial* property of Turing Machines is decidable.

That is, no *nontrivial* property of recognizable languages is decidable.

Consequences?

"That is, no nontrivial *property of recognizable languages is decidable."*

- ✓ $\mathrm{NE_{TM}} := \{ \langle M \rangle \mid M \text{ accepts at least one } w \in \Sigma^* \}$
- ✓ $\mathrm{E_{TM}} := \{ \langle M \rangle \mid L(M) = \emptyset \}$
- ✓ $\mathrm{ALL_{TM}} := \{ \langle M \rangle \mid L(M) = \Sigma^* \}$
- ✓ $\text{Accepts-s} := \{ \langle M \rangle \mid M \text{ accepts } s \}$
- ✓ $\text{Regular} := \{ \langle M \rangle \mid L(M) \text{ is regular} \}$

Proof (Sketch) of Rice's Theorem

Setup: we have a nontrivial property $P(L)$ of languages.

- ✓ WLOG, $\neg P(\emptyset)$.
- ✓ But *some* TM's language satisfies P. Pick a TM K such that $P(L(K))$.

Plan: find a mapping reduction from A_{TM} to P.

Given $\langle M, w \rangle$, construct M', which does the following:

1. Take input x. Save it an an auxiliary tape.
2. Run M on w.
3. If M accepts w, run K on x. Otherwise, reject.

Claim: M accepts w if and only if $L(M')$ has property P.

That is, if we had a P-checker, we could decide A_{TM}.

Limitations of Rice's Theorem

- ✓ It must be a functional property (i.e., *expressible as a property about languages*)
- ✓ It must be a nontrivial property among the languages accepted by TMs
- ✓ If so, the property is not decidable. But Rice's Theorem tells us nothing about whether the property is *semidecidable* or not.

CS 70 Grader Permanent Employment Theorem

The language

$$EQ_{TM} = \{ \langle M_1, M_2 \rangle \mid L(M_1) = L(M_2) \}$$

is not decidable.

The same holds for pairs of programs written in C++, Java, etc.

Full Employment Theoerem for Compiler Writers

Theorem

There is no perfect size-optimizing compiler.

Proof.

Any program that infinite loops without output could be identified, as it would reduce to a single loop instruction:

```
L1:   jmp L1
```

□

Perfect Garbage Collection is Undecidable

Java, Python, Haskell, Scheme, *etc.*, all rely on *garbage collection* to *deallocate* unused memory.

- At any point during execution, a piece of data is *live* it it will be used in the future, and otherwise *dead* or *garbage*.
- A *garbage collector* detects and deallocates garbage.
- *Perfect* garbage collection is undecidable.