$$(p \to q \to r) \to (p \to q) \to p \to r$$

1.	$p \to q \to r$	assumption
2.	$p \to q$	assumption
3.	p	assumption
4.	q	MP 2,3
5.	$q \to r$	MP 3,1
6.	r	MP 5,4
7.	$p \to r$	$\to$i 3-6
8.	$(p \to q) \to p \to r$	$\to$i 2-7
9.	$(p \to q \to r) \to (p \to q) \to p \to r$	$\to$i 1-8

Natural Deduction Rules

$$\frac{A \qquad B}{A \wedge B}\ \wedge i \qquad\qquad \frac{A \wedge B}{A}\ \wedge e1 \qquad\qquad \frac{A \wedge B}{B}\ \wedge e2$$

$$\frac{A}{A \vee B}\ \vee i1 \qquad\qquad \frac{B}{A \vee B}\ \vee i2 \qquad\qquad \frac{A \vee B \qquad \boxed{\begin{array}{c} A \\ \vdots \\ C \end{array}} \qquad \boxed{\begin{array}{c} B \\ \vdots \\ C \end{array}}}{C}\ \vee e$$

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{A \rightarrow B}\ \rightarrow i \qquad\qquad \frac{A \rightarrow B \qquad A}{B}\ \rightarrow e\ (\mathrm{MP})$$

$$\frac{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}{\neg A}\ \neg i \qquad\qquad \frac{\neg A \qquad A}{\bot}\ \neg e \qquad\qquad \frac{}{\top}\ \top i \qquad\qquad \frac{\bot}{A}\ \bot e$$

$$\frac{\neg\neg A}{A}\ \neg\neg e \qquad\qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \mathrm{PBC} \qquad\qquad \frac{}{A \vee \neg A}\ \mathrm{LEM}$$

Nonconstructive Proof

(repeated in the textbook)

Even the Greeks knew that $\sqrt{2}$ is irrational.

Consider $\sqrt{2}^{\sqrt{2}}$. Either it's rational, or it's not.

- If it is rational, we're done (with $a = b = \sqrt{2}$).

- If it's irrational, we're done (with $a = \sqrt{2}^{\sqrt{2}}$, $b = \sqrt{2}$).

In either case, one can find irrational a and b such that a^b is rational

QED.

Constructive Proof

(not in the textbook)

Put $a := \sqrt{2} = 2^{\frac{1}{2}}$. It's still irrational.

Put $b := \log_2 9 = 2 \log_2 3$. It's irrational too.

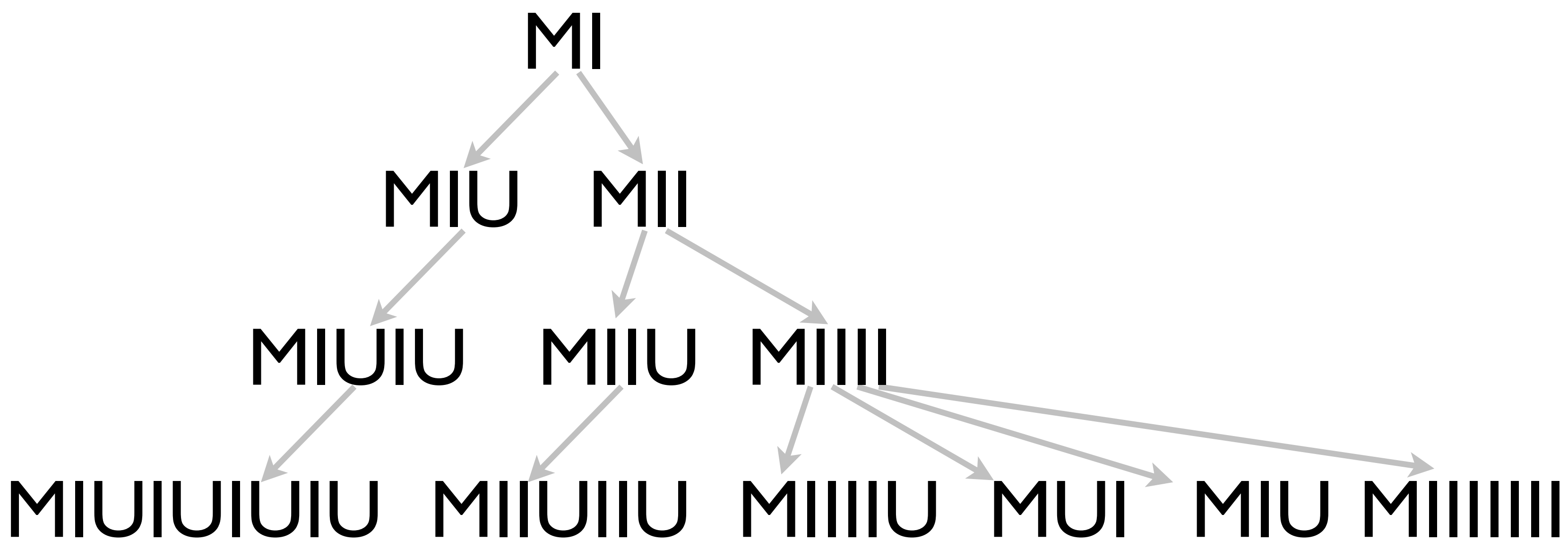
But $a^b = (2^{\frac{1}{2}})^{2 \log_2 3} = 2^{\left(\frac{1}{2} \times 2 \log_2 3\right)} = 2^{\log_2 3} = 3.$

QED.

Semantics of Natural Deduction

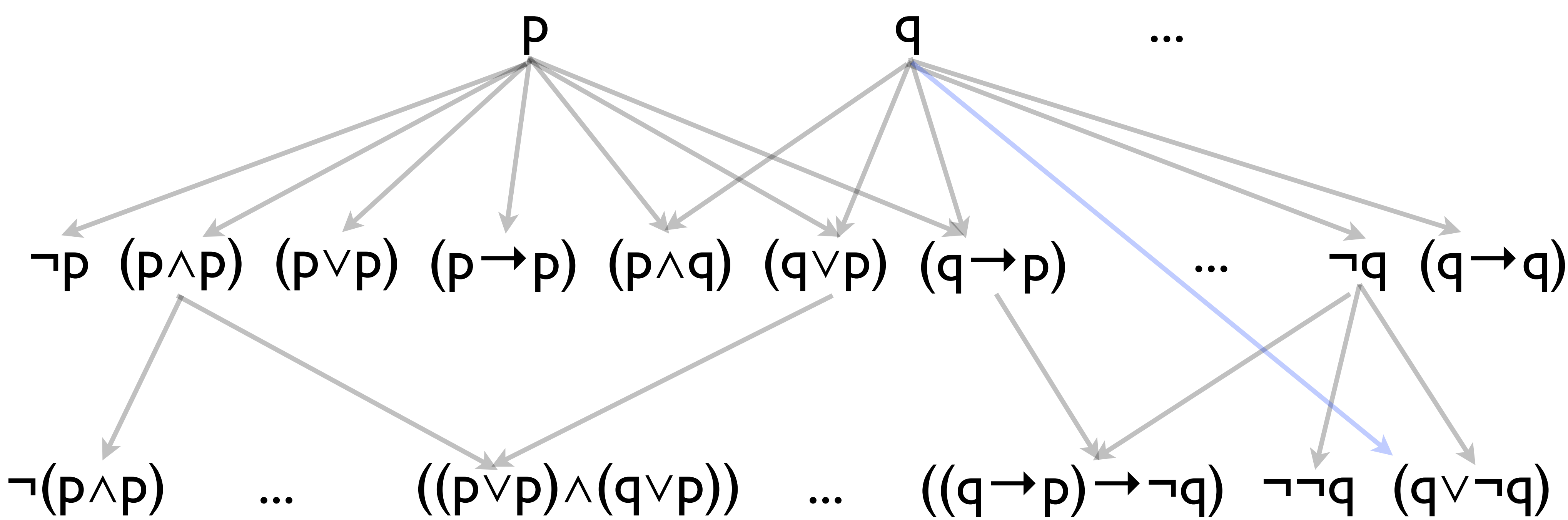
September 7, 2011
CS 81: Computability and Logic

MIU System



...etc...

Example: Propositional Formulas



...etc...

Natural Deduction Rules

$$\frac{A \qquad B}{A \wedge B}\ \wedge i \qquad\qquad \frac{A \wedge B}{A}\ \wedge e1 \qquad\qquad \frac{A \wedge B}{B}\ \wedge e2$$

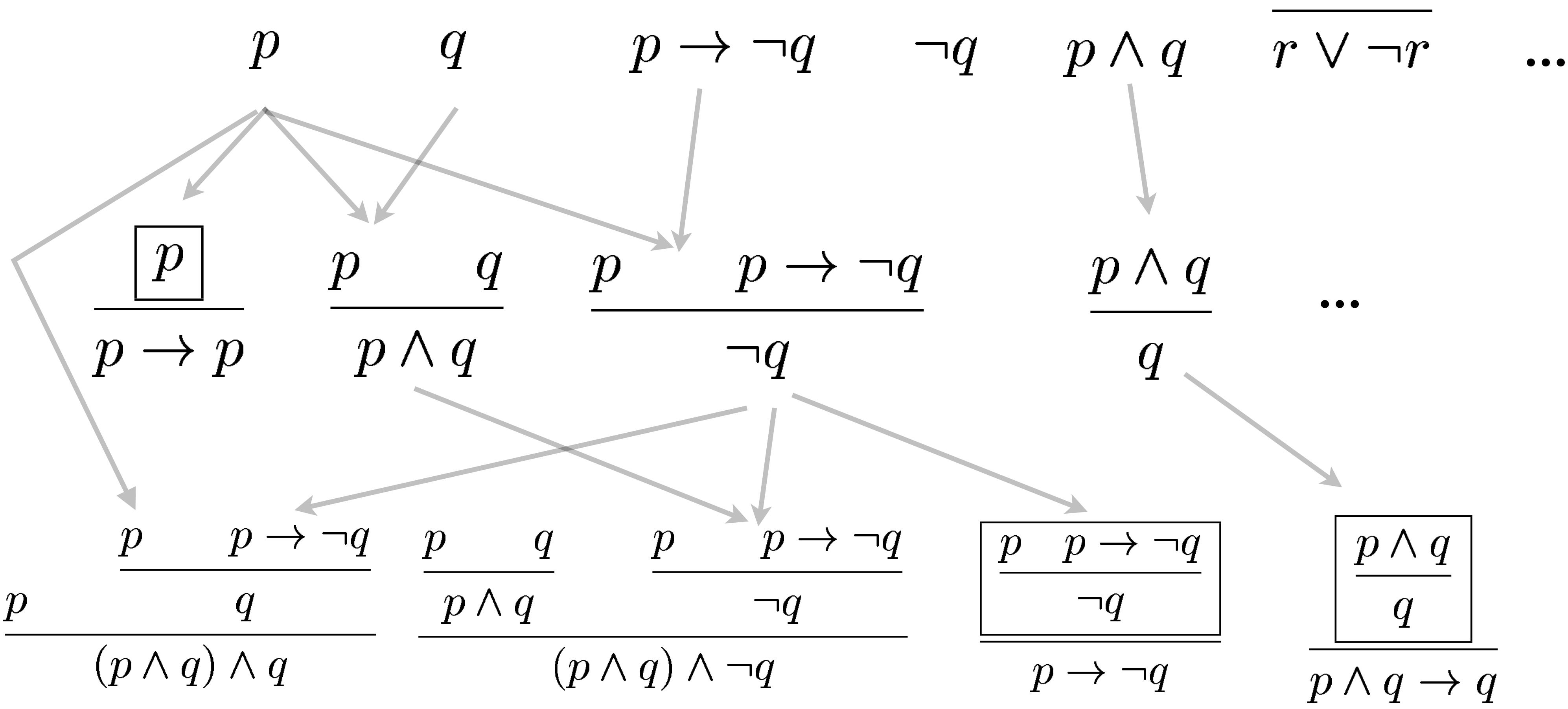
$$\frac{A}{A \vee B}\ \vee i1 \qquad\qquad \frac{B}{A \vee B}\ \vee i2 \qquad\qquad \frac{A \vee B \qquad \boxed{\begin{array}{c} A \\ \vdots \\ C \end{array}} \qquad \boxed{\begin{array}{c} B \\ \vdots \\ C \end{array}}}{C}\ \vee e$$

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{A \rightarrow B}\ \rightarrow i \qquad\qquad \frac{A \rightarrow B \qquad A}{B}\ \rightarrow e\ (\mathrm{MP})$$

$$\frac{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}{\neg A}\ \neg i \qquad\qquad \frac{\neg A \qquad A}{\bot}\ \neg e \qquad\qquad \frac{}{\top}\ \top i \qquad\qquad \frac{\bot}{A}\ \bot e$$

$$\frac{\neg\neg A}{A}\ \neg\neg e \qquad\qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\ \mathrm{PBC} \qquad\qquad \frac{}{A \vee \neg A}\ \mathrm{LEM}$$

Proofs *with Assumptions*



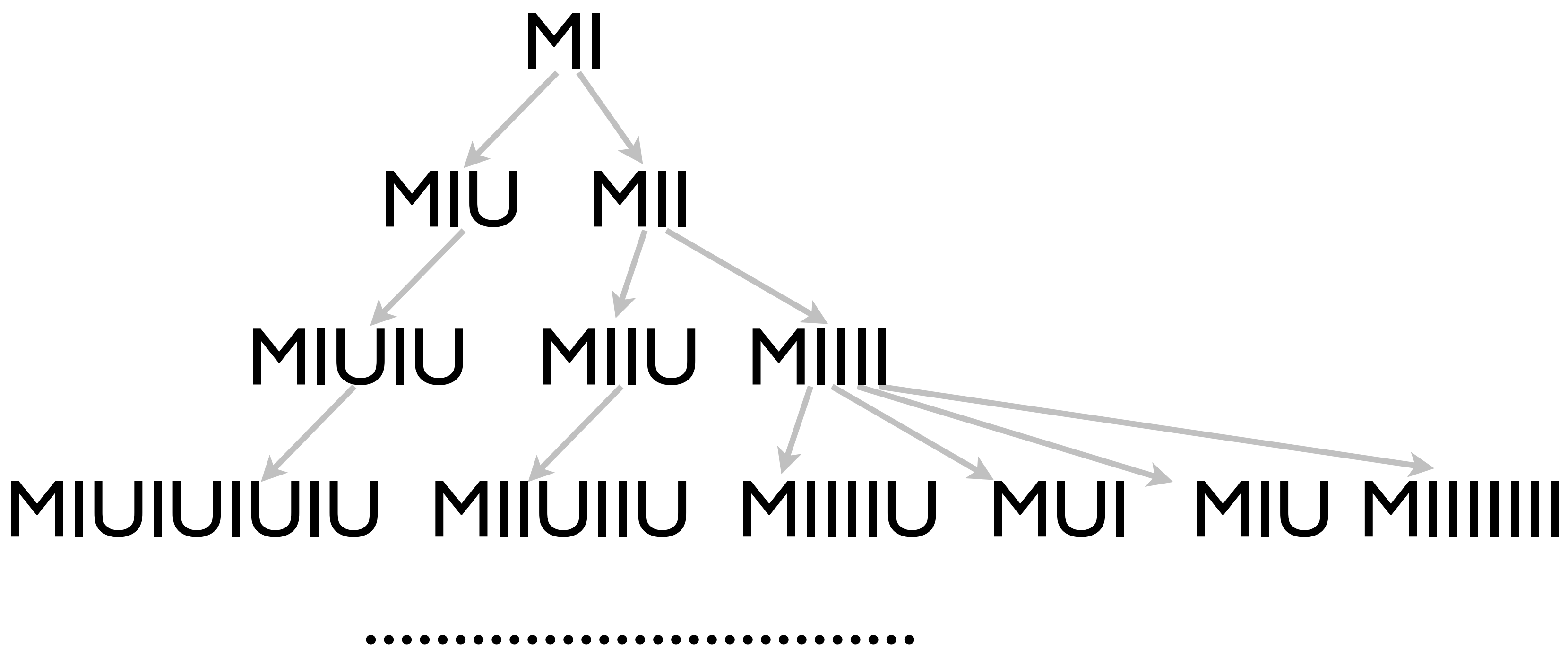
...etc...

Syntax vs. Semantics

We can always do "symbol manipulation"

But is there any meaning to our rules?

MIU System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Hofstadter's pq- System

Axiom Scheme (For any nonempty string of hyphens x):

$\vdash x$p-qx-

Rule of Inference (For hyphen strings x, y and z):

If $\vdash x$pyqz then $\vdash x$py-qz-.

Natural Deduction Rules

$$\frac{A \qquad B}{A \wedge B}\wedge_i \qquad \frac{A \wedge B}{A}\wedge_{e1} \qquad \frac{A \wedge B}{B}\wedge_{e2}$$

$$\frac{A}{A \vee B}\vee_{i1} \qquad \frac{B}{A \vee B}\vee_{i2} \qquad \frac{A \vee B \qquad \begin{matrix} A \\ \vdots \\ \hline C \end{matrix} \qquad \begin{matrix} B \\ \vdots \\ \hline C \end{matrix}}{C}\vee_e$$

$$\frac{\begin{matrix} A \\ \vdots \\ B \end{matrix}}{A \to B}\to_i \qquad \frac{A \to B \qquad A}{B}\to_e \text{(MP)}$$

$$\frac{\begin{matrix} A \\ \vdots \\ \bot \end{matrix}}{\neg A}\neg_i \qquad \frac{\neg A \qquad A}{\bot}\neg_e \qquad \frac{}{\top}\top_i \qquad \frac{\bot}{A}\bot_e$$

$$\frac{\neg\neg A}{A}\neg\neg_e \qquad \frac{\begin{matrix} \neg A \\ \vdots \\ \bot \end{matrix}}{A}\text{PBC} \qquad \frac{}{A \vee \neg A}\text{LEM}$$

Meanings of the Connectives

A	¬A
T	F
F	T

⊥
F

⊤
T

A	B	A∧B
F	F	F
F	T	F
T	F	F
T	T	T

A	B	A∨B
F	F	F
F	T	T
T	F	T
T	T	T

A	B	A→B
F	F	T
F	T	T
T	F	F
T	T	T

So: what is the meaning of "$(p \to (\neg p \wedge q))$" ?

Valuations/Models

$$v: \{ p, q, r, \ldots\} \rightarrow \{T,F\}.$$

E.g., $v_1(p) = T,\ v_1(q) = F,\ v_1(r) = T, \ldots$

or $v_2(p) = F,\ v_2(q) = F,\ v_2(r) = F, \ldots$

By "abuse of notation" I write

$$v(A)$$

to denote the *meaning* of a formula A.

Satisfiability

A formula is satisfiable if *some* valuation makes it true.

A formula is unsatisfiable if *no* valuation makes it true.

A formula is a tautology if *all* valuations make it true.

Equivalently, if its negation is ...

Semantic Entailment

$$B, C, D \models A$$

Whenever B, C, and D are <u>true</u>, so is A.

==

Every valuation that makes B, C, and D true also makes A true. (Other valuations might not)

Truth vs. Provability

A is true if B & C & D are:

$$B, C, D \models A$$

A is provable if B & C & D are:

$$B, C, D \vdash A$$

Soundness: Provability implies Truth

Completeness: Truth implies Provability

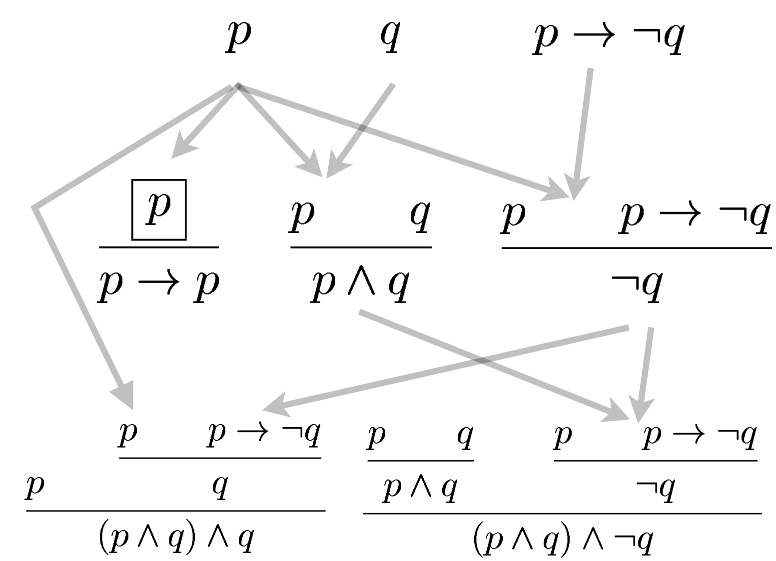
Soundness: If $\vdash$ then $\models$

Strategy: proof by induction on proof trees!

We want to show for every proof tree:

If a valuation makes its *assumptions* true

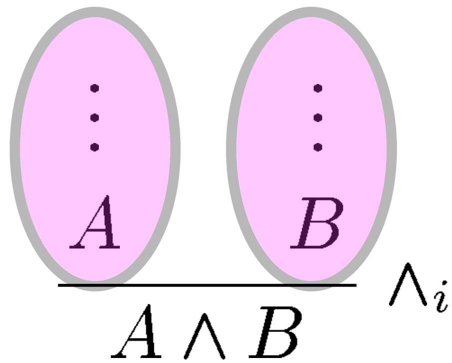
Then that valuation makes the *conclusion* true.



Case

To show: If a valuation makes the proof's *assumptions* true, then it also makes the *conclusion* true.

The proof has the form



Assume we have a valuation that makes the assumptions of this *proof-as-a-whole* true.

Case

To show: If a valuation makes the proof's *assumptions* true, then it also makes the *conclusion* true.

The proof has the form

$$\dfrac{\begin{array}{c}\vdots\\ A\end{array} \qquad \begin{array}{c}\vdots\\ A \to B\end{array}}{B}\ {\to}i$$

Assume we have a valuation that makes the assumptions of this *proof-as-a-whole* true.

...

To show: If a valuation makes the proof's *assumptions* true, then it also makes the *conclusion* true.

Case

The proof has the form

$$\frac{\begin{array}{c}A\\ \vdots\\ B\end{array}}{A \to B}$$

Assume we have a valuation that makes the assumptions of this *proof-as-a-whole* true.

...

Completeness: If $\vDash$ then $\vdash$

It suffices to prove "If $\vDash$ C then $\vdash$ C."

Assume $A_1, \ldots, A_n \vDash B$.

1. Show $\vDash A_1 \rightarrow \ldots A_n \rightarrow B$. Easy.

2. Show $\vdash A_1 \rightarrow \ldots A_n \rightarrow B$.

3. $A_1, \ldots, A_n \vdash B$. Easy.

Idea 1

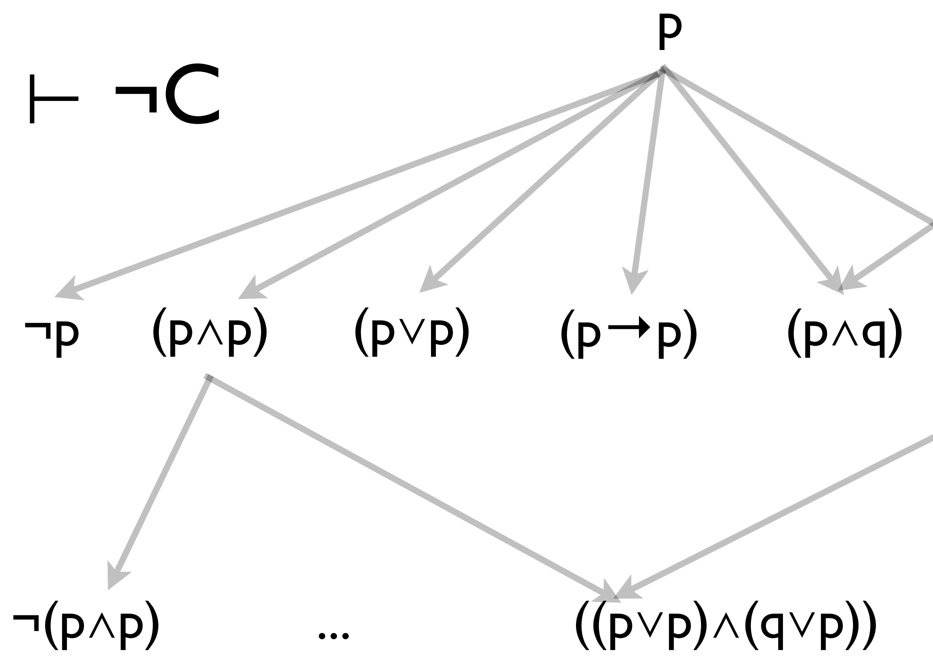
Note: we can generalize this idea for <u>any</u> valuation!

Suppose $v(p) = T \quad v(q) = F \quad v(r) = T$

Claim: For any formula C involving p,q,r:

if $v(C) = T$ then $p, \neg q, r \vdash C$

if $v(C) = F$ then $p, \neg q, r \vdash \neg C$



Proof: By induction on C!

Idea 2

Assume $\vDash C$.

Use LEM + proof by cases to consider <u>all</u> combinations

p vs. $\neg p$

q vs. $\neg q$

...all other variables in C...

For each of these 2^n cases, use Idea 1 to prove C.

Conclude $\vdash C$.

Consequences

There's an algorithm for determining provability (or non-provability) in classical propositional logic.

There's even an algorithm for creating a (horrible) proof when the formula is provable.

Everything you already know about truth tables and boolean logic still applies.