\[(p \rightarrow q \rightarrow r) \rightarrow (p \rightarrow q) \rightarrow p \rightarrow r\]

1. \(p \rightarrow q \rightarrow r\)  
   assumption
2. \(p \rightarrow q\)  
   assumption
3. \(p\)  
   assumption
4. \(q\)  
   MP 2,3
5. \(q \rightarrow r\)  
   MP 3,1
6. \(r\)  
   MP 5,4
7. \(p \rightarrow r\)  
   \(\rightarrow i\) 3-6
8. \((p \rightarrow q) \rightarrow p \rightarrow r\)  
   \(\rightarrow i\) 2-7
9. \((p \rightarrow q \rightarrow r) \rightarrow (p \rightarrow q) \rightarrow p \rightarrow r\)  
   \(\rightarrow i\) 1-8
Natural Deduction Rules

\[ \frac{A \quad B}{A \land B} \quad \land i \]

\[ \frac{A \land B}{A} \quad \land e_1 \]

\[ \frac{A \land B}{B} \quad \land e_2 \]

\[ \frac{A \lor B}{A \lor B} \quad \lor i_1 \]

\[ \frac{B}{A \lor B} \quad \lor i_2 \]

\[ \frac{A \lor B \quad A \quad C \quad B}{C} \quad \lor e \]

\[ \frac{A \rightarrow B}{A \rightarrow B} \quad \rightarrow i \]

\[ \frac{A \rightarrow B}{B} \quad \rightarrow e \quad (MP) \]

\[ \frac{A}{\top} \quad \neg i \]

\[ \frac{\neg A \quad A}{\bot} \quad \neg e \]

\[ \frac{\bot}{A} \quad \bot i \]

\[ \frac{\bot}{\bot} \quad \bot e \]

\[ \frac{\neg A}{\neg A} \quad \neg e \]

\[ \frac{\bot}{A} \quad \text{PBC} \]

\[ \frac{A \lor \neg A}{A} \quad \text{LEM} \]
Nonconstructive Proof
(repeated in the textbook)

Even the Greeks knew that \( \sqrt{2} \) is irrational.

Consider \( \sqrt{2}^{\sqrt{2}} \). Either it’s rational, or it’s not.

- If it is rational, we’re done (with \( a = b = \sqrt{2} \)).

- If it’s irrational, we’re done (with \( a = \sqrt{2}^{\sqrt{2}} \), \( b = \sqrt{2} \)).

In either case, one can find irrational \( a \) and \( b \) such that \( a^b \) is rational.

QED.
Constructive Proof
(not in the textbook)

Put $a := \sqrt{2} = 2^{\frac{1}{2}}$. It’s still irrational.

Put $b := \log_2 9 = 2 \log_2 3$. It’s irrational too.

But $a^b = \left(2^{\frac{1}{2}}\right)^{2 \log_2 3} = 2^{\left(\frac{1}{2} \times 2 \log_2 3\right)} = 2^{\log_2 3} = 3$.

QED.
Semantics of Natural Deduction

September 7, 2011
CS 81: Computability and Logic
MIU System

...etc...
Example: Propositional Formulas

\[
\neg p \quad (p \wedge p) \quad (p \lor p) \quad (p \rightarrow p) \quad (p \wedge q) \quad (q \lor p) \quad (q \rightarrow p) \quad \ldots \quad \neg q \quad (q \rightarrow q)
\]

\[
\neg (p \wedge p) \quad \ldots \quad ((p \lor p) \wedge (q \lor p)) \quad \ldots \quad ((q \rightarrow p) \rightarrow \neg q) \quad \neg p \quad (q \lor q)
\]

\ldots\text{etc}\ldots
Natural Deduction Rules

\[ \frac{A \quad B}{A \land B} \land i \]
\[ \frac{A \land B}{A} \land e_1 \]
\[ \frac{A \land B}{B} \land e_2 \]
\[ \frac{A}{A \lor B} \lor i \]
\[ \frac{B}{A \lor B} \lor i_2 \]
\[ \frac{A \lor B}{C} \lor e \]
\[ \frac{A}{A \rightarrow B} \rightarrow i \]
\[ \frac{B}{A \rightarrow B} \rightarrow e \ (MP) \]
\[ \frac{\bot}{\neg A} \neg i \]
\[ \frac{\neg A}{\bot} \neg e \]
\[ \frac{\bot}{A} \bot i \]
\[ \frac{A}{\bot} \bot e \]
\[ \frac{\neg A}{\neg \neg \neg A} \neg \neg \neg e \]
\[ \frac{A}{PBC} \]
\[ \frac{A \lor \neg A}{\bot} \text{LEM} \]
Proofs with Assumptions

...etc...
Syntax vs. Semantics

We can always do “symbol manipulation”

But is there any meaning to our rules?
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Hofstadter’s pq- System

Axiom Scheme (For any nonempty string of hyphens $x$):

$$\vdash xp-qx-$$

Rule of Inference (For hyphen strings $x$, $y$ and $z$):

If $\vdash xpyqz$ then $\vdash xpy-qz-$.
Natural Deduction Rules

\[
\frac{A}{A \land B} \quad \frac{B}{A \land B} \quad \frac{A \land B}{A} \quad \frac{A \land B}{B} \\
\land_i \quad \land_e1 \quad \land_e2
\]

\[
\frac{A}{A \lor B} \quad \frac{B}{A \lor B} \quad \frac{A \lor B}{C} \quad \frac{A \lor B}{C} \quad \frac{A \lor B}{C} \\
\lor_i1 \quad \lor_i2 \quad \lor_e
\]

\[
\frac{A}{A \rightarrow B} \quad \frac{B}{A \rightarrow B} \quad \frac{A \rightarrow B \ A}{B} \quad \frac{A \rightarrow B \ A}{B} \quad \frac{A \rightarrow B \ A}{B} \\
\rightarrow_i \quad \rightarrow_e (MP)
\]

\[
\frac{A}{\bot} \quad \frac{\neg A \quad A}{\bot} \quad \frac{\bot}{\top} \quad \frac{\top}{A} \quad \frac{\bot}{A} \quad \frac{\bot}{A} \quad \frac{\bot}{A} \\
\neg_i \quad \neg_e \quad \top_i \quad \bot_e
\]

\[
\frac{\neg \neg A \quad \neg \neg e}{A} \quad \frac{\bot}{PBC} \quad \frac{A \lor \neg A}{\text{LEM}}
\]
Meanings of the Connectives

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<tr>
<th>A</th>
<th>¬A</th>
<th>F</th>
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<thead>
<tr>
<th>A</th>
<th>B</th>
<th>A ∧ B</th>
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So: what is the meaning of “(p → (¬p ∧ q))”? 
Valuations/Models

\[ v: \{ p, q, r, \ldots \} \rightarrow \{ T, F \}. \]

E.g., \( v_1(p) = T, \, v_1(q) = F, \, v_1(r) = T, \ldots \)

or \( v_2(p) = F, \, v_2(q) = F, \, v_2(r) = F, \ldots \)

By “abuse of notation” I write

\[ v(A) \]

to denote the meaning of a formula \( A \).
Satisfiability

A formula is **satisfiable** if *some* valuation makes it true.

A formula is **unsatisfiable** if *no* valuation makes it true.

A formula is a **tautology** if *all* valuations make it true.

Equivalently, if its negation is ...
Semantic Entailment

\[ B, C, D \models A \]

Whenever B, C, and D are true, so is A.

==

Every valuation that makes B, C, and D true also makes A true. (Other valuations might not)
Truth vs. Provability

A is true if $B \land C \land D$ are:

$B, C, D \models A$

A is provable if $B \land C \land D$ are:

$B, C, D \vdash A$

Soundness: Provability implies Truth

Completeness: Truth implies Provability
Soundness: If ⊢ then ⊨

Strategy: proof by induction on proof trees!

We want to show for every proof tree:

If a valuation makes its assumptions true

Then that valuation makes the conclusion true.
Case

The proof has the form

\[ \vdots \quad \vdots \]
\[
\begin{array}{c}
A \\
A \land B
\end{array}
\quad \quad
\begin{array}{c}
B \\
\land_i
\end{array}
\]

Assume we have a valuation that makes the assumptions of this proof-as-a-whole true.

**To show:** If a valuation makes the proof’s assumptions true, then it also makes the conclusion true.
Case

The proof has the form

\[ \frac{A \quad A \rightarrow B}{B} \rightarrow i \]

Assume we have a valuation that makes the assumptions of this proof-as-a-whole true.

To show: If a valuation makes the proof’s assumptions true, then it also makes the conclusion true.

...
Case

The proof has the form

\[ A \rightarrow B \]

Assume we have a valuation that makes the assumptions of this *proof-as-a-whole* true.

...
Completeness: If $\models$ then $\vdash$

It suffices to prove “If $\models C$ then $\vdash C$.”

Assume $\mathbf{A}_1, \ldots, \mathbf{A}_n \models \mathbf{B}$.

1. Show $\models \mathbf{A}_1 \rightarrow \ldots \mathbf{A}_n \rightarrow \mathbf{B}$. Easy.

2. Show $\vdash \mathbf{A}_1 \rightarrow \ldots \mathbf{A}_n \rightarrow \mathbf{B}$.

3. $\mathbf{A}_1, \ldots, \mathbf{A}_n \vdash \mathbf{B}$. Easy.
Idea 1

Suppose \( v(p) = T \), \( v(q) = F \), \( v(r) = T \)

\[ \begin{align*}
\text{Claim: For any formula } C \text{ involving } p, q, r: \\
\text{if } v(C) = T \text{ then } p, \neg q, r \vdash C \\
\text{if } v(C) = F \text{ then } p, \neg q, r \vdash \neg C
\end{align*} \]

Proof: By induction on \( C \)!

Note: we can generalize this idea for any valuation!
Idea 2

Assume $\models C$.

Use LEM + proof by cases to consider all combinations

- $p$ vs. $\neg p$
- $q$ vs. $\neg q$
- ...all other variables in $C$...

For each of these $2^n$ cases, use Idea 1 to prove $C$.

Conclude $\vdash C$. 
Consequences

There’s an algorithm for determining provability (or non-provability) in classical propositional logic.

There’s even an algorithm for creating a (horrible) proof when the formula is provable.

Everything you already know about truth tables and boolean logic still applies.