Worksheet: Predicate Calculus Proofs

1. Your math professor asked you to prove (under certain assumptions which I won’t repeat here) that \( \forall m > 5, f(m) < n \). Should you get full credit for a proof that:

   (a) \( \forall m. (m > 5) \land f(m) < n \) ?

   (b) \( \forall m. (m > 5) \rightarrow f(m) < n \) ?

   (c) \( (\forall m. m > 5) \rightarrow f(m) < n \) ?

   (d) \( \forall k > 5. f(k) < n \) ?

   (e) \( \forall m > 5. f(m) < k \) ?

2. Recall the \( A[t/x] \) is the book’s notation for substitution, and refers to the result of replacing free occurrences of the variable \( x \) in \( A \) by the term \( t \). What is:

   (a) \( (\text{Even}(y) \rightarrow \text{Odd}(1 + y \cdot 2)) [2+2 / y] \)

   (b) \( (z > 3 \lor (\forall z. z^2 > z + y + w)) [7 / z] \)

   (c) \( (z > 3 \lor (\forall z. z^2 > z + y + w)) [2+z / y] \)

3. One “specific instance” (of many) of the \( \land \iota \) rule is:

\[
\forall z. g(z) < 7 \quad f(y) > 5. \\
(\forall z. g(z) < 7) \land f(y) > 5.
\]

Show specific instances of \( \forall \iota, \forall e, \exists \iota, \) and \( \exists e. \)
4. Prof. Karp kindly provided an excerpt from his lecture notes (see the last page).

(a) Look at the Definition 4 (of “limit”). Rewrite the definition of \( \lim_{x \to x_0} f(x) = L \) in the formal logic notation of our Predicate Calculus.

\[ \forall \epsilon, \epsilon > 0 \rightarrow \exists \delta. (\delta > 0) \land (\forall x. (0 < |x - x_0|) \land (|x - x_0| < \delta) \rightarrow |f(x) - L| < \epsilon) \]

(b) Consider the proof just above Remark 7. What rules of natural deduction correspond to (explicit or implicit) steps of this proof?

\[ \forall i, \exists i, \rightarrow i \]

5. Prove \( t_1 = t_2 \vdash t_2 = t_1 \)

1. \( t_1 = t_2 \) Assumption
2. \( t_1 = t_1 \) = i
3. \( t_2 = t_1 \) = e2, 1

6. Prove \( t_1 = t_2, t_2 = t_3 \vdash t_1 = t_3 \)

1. \( t_1 = t_2 \) Assumption
2. \( t_2 = t_3 \) Assumption
3. \( t_1 = t_3 \) = e2, 1
7. Prove \( \neg \exists x. (B(x) \land G(x)), \quad \forall x. (D(x) \rightarrow B(x)) \vdash \neg \exists x. (D(x) \land G(x)) \)

1. \( \neg \exists x. (B(x) \land G(x)) \)  Assumption
2. \( \forall x. (D(x) \rightarrow B(x)) \)  Assumption
3. \( \exists x. (D(x) \land G(x)) \)  Assumption

\[ \begin{align*}
\text{x}_0 & \quad \text{D(x}_0) & \text{\lor e 4} \\
\text{x} & \quad \text{G(x}_0) & \text{\lor e 4} \\
\text{6} & \quad \text{D(x}_0) \rightarrow \text{B(x}_0) & \text{\lor e 2} \\
\text{7} & \quad \text{B(x}_0) & \text{MP 7, 5} \\
\text{8} & \quad \text{B(x}_0) \land \text{G(x}_0) & \text{\lor i 8, 6} \\
\text{9} & \quad \exists x. \text{B(x) \land G(x)} & \text{\lor i 9} \\
\text{10} & \quad \bot & \text{\lor e 1, 9} \\
\end{align*} \]

\[ \begin{align*}
\text{11} & \quad \bot & \text{\lor e 3, 4-10} \\
\text{12} & \quad \neg \exists x. (D(x) \land G(x)) & \text{\lor i 3-11} \\
\end{align*} \]

8. Prove \( \forall x. P(x) \vdash \neg \exists x. \neg P(x) \).

1. \( \forall x. P(x) \)  Assumption
2. \( \exists x. \neg P(x) \)  Assumption

\[ \begin{align*}
\text{x}_0 & \quad \neg P(x}_0) & \text{\lor e 4} \\
\text{x} & \quad \neg P(x}_0) & \text{\lor e 4} \\
\text{4} & \quad \neg P(x}_0) & \text{\lor e 4, 3} \\
\text{5} & \quad \bot & \text{\lor e 4, 3} \\
\text{6} & \quad \bot & \text{\lor e 2, 3-5} \\
\text{7} & \quad \neg \exists x. \neg P(x) &
\end{align*} \]
9. Prove $\exists z. \forall w. P(z, w) \vdash \forall y. \exists x. P(x, y)$

1  $\exists z. \forall w. P(z, w)$  Assumption
2  $\forall w. P(\mathbb{y}_0, w)$  Assumption
3  $P(\mathbb{z}_0, \mathbb{y}_0)$  $\forall e$ 3
4  $\exists x. P(\mathbb{x}, \mathbb{y}_0)$  $\exists i$ 4
5  $\exists x. P(\mathbb{x}, \mathbb{y}_0)$  $\exists e$ 1, 3–5
6  $\forall y. \exists x. P(x, y)$  $\forall i$ 2–6

10. Prove $\forall x. P(x) \vdash \exists x. P(x)$

1  $\forall x. P(x)$  Assumption
2  $P(z)$  $\forall e$ 1
3  $\exists x. P(x)$  $\exists i$ 2

11. What is wrong with this proof of $\exists x. P(x) \vdash \forall x. Q(x)$?

1  $\exists x. P(x)$  Assumption
2  $\forall x. (P(x) \rightarrow Q(x))$  Assumption
3  $P(\mathbb{x}_0)$  Assumption
4  $P(\mathbb{x}_0)$  $\forall e$ 2
5  $P(\mathbb{x}_0) \rightarrow Q(\mathbb{x}_0)$  $\forall e$ 2
6  $Q(\mathbb{x}_0)$  MP 5,4
7  $Q(\mathbb{x}_0)$  $\exists e$ 1, 4–6
8  $\forall y. Q(y)$  $\forall i$ 3–7
**Definition 4.** Let \( f(x) \) be a real valued function, let \( x_0 \in \mathbb{R} \) be a point, not necessarily in the domain of \( f \), and let \( L \) be a real number. The limit of \( f(x) \) as \( x \) approaches \( x_0 \) is equal to \( L \) if and only if for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that
\[
0 < |x - x_0| < \delta \text{ implies } |f(x) - L| < \epsilon.
\]
In this case we write
\[
\lim_{x \to x_0} f(x) = L.
\]

**Remark 5.** First, note the correspondence between this definition and our intuitive first approximation. Second, note the geometric nature of this definition. Lastly, note the importance of distinction between \( x \) and \( x_0 \); the distance between the two is required to be strictly positive.

**Example 6.** Prove \( \lim_{x \to 1} 8x - 3 = 5 \).

Before we begin our proof, let's first study the problem. We must show for each \( \epsilon > 0 \) there is \( \delta > 0 \) such that \( 0 < |x - 1| < \delta \text{ implies } |(8x - 3) - 5| < \epsilon \).

So, we perform exploratory computations.
\[
|(8x - 3) - 5| = |8x - 8| < \epsilon \\
\iff 8|x - 1| < \epsilon \\
\iff |x - 1| < \epsilon/8
\]

Our quest to discover \( \delta \) seems to be complete. We have found a number, namely \( \delta = \epsilon/8 \), such that \( |x - 1| < \delta \text{ implies } |(8x - 3) - 5| < \epsilon \). We now formalize this proof.

**Proof.** Let \( \epsilon > 0 \) be given. Choose \( \delta = \epsilon/8 \). Then if
\[
0 < |x - 1| < \delta = \epsilon/8
\]
Then
\[
|(8x - 3) - 5| = |8x - 8| = 8|x - 1| < 8\delta = 8 \cdot \epsilon/8 = \epsilon.
\]
Therefore \( |f(x) - L| < \epsilon \), as desired. \( \square \)

**Remark 7.** Note the double edged practice of covering one's tracks displayed in this proof. As written, the choice of \( \delta \) in the above proof may appear initially unmotivated. On the other hand, the proof is rather concise without inclusion of exploratory calculations.