Worksheet: Predicate Calculus Proofs

1. Your math professor asked you to prove (under certain assumptions which I won't repeat here) that $\forall m > 5.\, f(m) < n$. Should you get full credit for a proof that:

   (a) $\forall m.\ (m > 5) \wedge f(m) < n$?

   (b) $\forall m.\ (m > 5) \to f(m) < n$?

   (c) $(\forall m.\, m > 5) \to f(m) < n$?

   (d) $\forall k > 5.\, f(k) < n$?

   (e) $\forall m > 5.\, f(m) < k$?

2. Recall the $A[t/x]$ is the book's notation for substitution, and refers to *the result of* replacing free occurrences of the variable x in A by the term t. What is:

   (a) $(\text{Even}(y) \to \text{Odd}(1 + y * 2))\,[2{+}2\,/\,y]$

   (b) $(z > 3 \vee (\forall z.\, z^2 > z + y + w))\,[7\,/\,z]$

   (c) $(z > 3 \vee (\forall z.\, z^2 > z + y + w))\,[2{*}z\,/\,y]$

3. One "specific instance" (of many) of the $\wedge i$ rule is:

$$\frac{\forall z. g(z) < 7 \qquad f(y) > 5}{(\forall z. g(z) < 7)\ \wedge\ f(y) > 5}.$$

   Show specific instances of $\forall i$, $\forall e$, $\exists i$, and $\exists e$.

4. Prof. Karp kindly provided an excerpt from his lecture notes (see the last page).

   (a) Look at the Definition 4 (of "limit"). Rewrite the definition of $\lim_{x \to x_0} f(x) = L$ in the formal logic notation of our Predicate Calculus.

   $$\forall \epsilon.\ \epsilon > 0 \to \exists \delta.\ (\delta > 0) \wedge \big(\forall x.\ (0 < |x - x_0|) \wedge (|x - x_0| < \delta) \to |f(x) - L| < \epsilon\big)$$

   (b) Consider the *proof* just above Remark 7. What rules of natural deduction correspond to (explicit or implicit) steps of this proof?

   $\forall i$, $\exists i$, $\to i$

5. Prove $t_1 = t_2 \vdash t_2 = t_1$

1	$t_1 = t_2$	Assumption
2	$t_1 = t_1$	$= i$
3	$t_2 = t_1$	$= e$2, 1

6. Prove $t_1 = t_2, t_2 = t_3 \vdash t_1 = t_3$

1	$t_1 = t_2$	Assumption
2	$t_2 = t_3$	Assumption
3	$t_1 = t_3$	$= e$2, 1

7. Prove $\neg\exists x.\,(B(x) \wedge G(x)),\quad \forall x.\,(D(x) \to B(x)) \quad\vdash\quad \neg\exists x.\,(D(x) \wedge G(x))$

	1	$\neg\exists x.\,(B(x) \wedge G(x))$	Assumption
	2	$\forall x.\,(D(x) \to B(x))$	Assumption
	3	$\exists x.\,(D(x) \wedge G(x))$	Assumption
x_0	4	$D(x_0)$	$\wedge$e 4
	5	$G(x_0)$	$\wedge$e 4
	6	$D(x_0) \to B(x_0)$	$\forall$e 2
	7	$B(x_0)$	MP 7, 5
	8	$B(x_0) \wedge G(x_0)$	$\wedge$i 8, 6
	9	$\exists x.\,B(x) \wedge G(x)$	$\exists$i 9
	10	$\bot$	$\neg$e 1, 9
	11	$\bot$	$\exists$e 3, 4–10
	12	$\neg\exists x.\,(D(x) \wedge G(x))$	$\neg$i 3–11

8. Prove $\forall x.\,P(x) \vdash \neg\exists x.\,\neg P(x)$.

	1	$\forall x.\,P(x)$	Assumption
	2	$\exists x.\,\neg P(x)$	Assumption
x_0	3	$\neg P(x_0)$	Assumption
	4	$P(x_0)$	$\forall$e 1, 3
	5	$\bot$	$\neg$e 4, 3
	6	$\bot$	$\exists$e 2, 3–5
	7	$\neg\exists x.\,\neg P(x)$	

9. Prove $\exists z.\, \forall w.\, P(z, w) \vdash \forall y.\, \exists x.\, P(x, y)$

	1	$\exists z.\, \forall w.\, P(z, w)$	Assumption
y_0	2		
z_0	3	$\forall w.\, P(z_0, w)$	Assumption
	4	$P(z_0, y_0)$	$\forall$e 3
	5	$\exists x. P(x, y_0)$	$\exists$i 4
	6	$\exists x. P(x, y_0)$	$\exists$e 1, 3–5
	7	$\forall y.\, \exists x. P(x, y)$	$\forall$i 2–6

10. Prove $\forall x.\, P(x) \quad \vdash \quad \exists x.\, P(x)$

1	$\forall x.\, P(x)$	Assumption
2	$P(z)$	$\forall$e 1
3	$\exists x.\, P(x)$	$\exists$i 2

11. What is wrong with this proof of $\exists x. P(x) \vdash \forall x. Q(x)$?

	1	$\exists x.\, P(x)$	Assumption
	2	$\forall x.\, (P(x) \to Q(x))$	Assumption
x_0	3		
x_0	4	$P(x_0)$	Assumption
	5	$P(x_0) \to Q(x_0)$	$\forall$e 2
	6	$Q(x_0)$	MP 5,4
	7	$Q(x_0)$	$\exists$e 1, 4–6
	8	$\forall y.\, Q(y)$	$\forall$i 3–7

Definition 4. *Let $f(x)$ be a real valued function, let $x_0 \in \mathbb{R}$ be a point, not necessarily in the domain of f, and let L be a real number. The* limit of $f(x)$ as x approaches x_0 is equal to L *if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$0 < |x - x_0| < \delta \text{ implies } |f(x) - L| < \epsilon.$$

In this case we write

$$\lim_{x \to x_0} f(x) = L.$$

Remark 5. First, note the correspondence between this definition and our intuitive first approximation. Second, note the geometric nature of this definition. Lastly, note the importance of distinction between x and x_0; the distance between the two is required to be strictly positive.

Example 6. *Prove* $\lim_{x \to 1} 8x - 3 = 5$.

Before we begin our proof, lets first study the problem. We must show for each $\epsilon > 0$ there is $\delta > 0$ such that $0 < |x - 1| < \delta$ implies $|(8x - 3) - 5| < \epsilon$.

So, we perform exploratory computations.

$$\begin{aligned} |(8x-3)-5| = |8x-8| &< \epsilon \\ \iff 8|x-1| &< \epsilon \\ \iff |x-1| &< \epsilon/8 \end{aligned}$$

Our quest to discover δ seems to be complete. We have found a number, namely $\delta = \epsilon/8$, such that $|x - 1| < \delta$ implies $|(8x - 3) - 5| < \epsilon$. We now formalize this proof.

Proof. Let $\epsilon > 0$ be given. Choose $\delta = \epsilon/8$. Then if

$$0 < |x - 1| < \delta = \epsilon/8$$

Then

$$|(8x - 3) - 5| = |8x - 8| = 8|x - 1| < 8\delta = 8 \cdot \epsilon/8 = \epsilon.$$

Therefore $|f(x) - L| < \epsilon$, as desired. □

Remark 7. Note the double edged practice of covering one's tracks displayed in this proof. As written, the choice of δ in the above proof may appear initially unmotivated. On the other hand, the proof is rather concise without inclusion of exploratory calculations.