# Worksheet: Natural Deduction to Sequent Calculus and Resolution

## 1 Natural Deduction in "Sequent Style" For comparison only!

There is an alternative way of organizing the natural deduction rules you already know; it uses $\vdash$ as part of the syntax of the proof—rather than as a metalevel statement about provability—to make assumptions explicit. *This version of natural deduction is presented only for comparison, and will never show up again in this class.* (It does have very cool connections to PL theory, though.)

$$\frac{\ldots \vdash A \qquad \ldots \vdash B}{\ldots \vdash A \wedge B}\wedge i \qquad \frac{\ldots \vdash A \wedge B}{\ldots \vdash A}\wedge e1 \qquad \frac{\ldots \vdash A \wedge B}{\ldots \vdash B}\wedge e2 \qquad \frac{}{\ldots \vdash \top}\top i$$

$$\frac{\ldots \vdash A}{\ldots \vdash A \vee B}\vee i1 \qquad \frac{\ldots \vdash B}{\ldots \vdash A \vee B}\vee i2 \qquad \frac{\ldots \vdash A \vee B \qquad \ldots, A \vdash C \qquad \ldots, B \vdash C}{\ldots \vdash C}\vee e$$

$$\frac{\ldots, A \vdash \bot}{\ldots \vdash \neg A}\neg i \qquad \frac{\ldots \vdash \neg A \qquad \ldots \vdash A}{\ldots \vdash \bot}\neg e \qquad \frac{\ldots \vdash \bot}{\ldots \vdash A}\bot e$$

$$\frac{\ldots, A \vdash B}{\ldots \vdash A \to B}\to i \qquad \frac{\ldots \vdash A \to B \qquad \ldots \vdash A}{\ldots \vdash B}\text{MP}$$

$$\frac{\ldots \vdash \neg\neg A}{\ldots \vdash A}\neg\neg e \qquad \frac{\ldots, \neg A \vdash \bot}{\ldots \vdash A}\text{RAA} \qquad \frac{}{\ldots \vdash A \vee \neg A}\text{LEM}$$

$$\frac{\ldots \vdash P(x_0) \qquad x_0 \text{ not free in } \ldots}{\ldots \vdash \forall x.\, P(x)}\forall i \qquad \frac{\ldots \vdash \forall x.\, P(x)}{\ldots \vdash P(t)}\forall e$$

$$\frac{\ldots \vdash P(t)}{\ldots \vdash \exists x.\, P(x)}\exists i \qquad \frac{\ldots \vdash \exists x.\, P(x) \qquad \ldots, P(x_0) \vdash C \qquad x_0 \text{ not free in } \ldots}{C}\exists e$$

$$\frac{}{\ldots, A, \cdots \vdash A}\textsc{Copy}$$

## 2 Sequent Calculus Rules

Yet another way of organizing logical rules (besides introduction vs. elimination, or splitting vs. stacking) is left vs. right. This leads us to Gentzen's (classical) *Sequent Calculus*.

Here we allow *sequences* of formulas on both sides. The intuition is that $A, B, C \vdash E, F, G$ means "under the assumptions $A$ and $B$ and $C$, at least one of $E$ or $F$ or $G$ is true."

**Logical Rules**

$$\frac{}{A \vdash A}\ \textsc{Assump} \qquad \frac{}{\bot \vdash}\ \bot L \qquad \frac{}{\vdash \top}\ \top R$$

$$\frac{\ldots, A, B \vdash ---}{\ldots, A \wedge B \vdash ---}\ \wedge L \qquad \frac{\ldots \vdash A, --- \qquad \ldots \vdash B, ---}{\ldots \vdash A \wedge B, ---}\ \wedge R$$

$$\frac{\ldots, A \vdash --- \qquad \ldots, B \vdash ---}{\ldots, A \vee B \vdash ---}\ \vee L \qquad \frac{\ldots \vdash A, B, ---}{\ldots \vdash A \vee B, ---}\ \vee R$$

$$\frac{\ldots \vdash A, --- \qquad \ldots, B \vdash ---}{\ldots, A \to B \vdash ---}\ \to L \qquad \frac{\ldots, A \vdash B, ---}{\ldots \vdash A \to B, ---}\ \to R$$

$$\frac{\ldots \vdash A, ---}{\ldots, \neg A \vdash ---}\ \neg L \qquad \frac{\ldots, A \vdash ---}{\ldots \vdash \neg A, ---}\ \neg R$$

**Structural Rules (ignore if you treat $\ldots$ and $---$ as sets)**

$$\frac{\ldots \vdash ---}{\ldots, A \vdash ---}\ \textsc{Weak-L} \qquad \frac{\ldots \vdash ---}{\ldots \vdash A, ---}\ \textsc{Weak-R}$$

$$\frac{\ldots, A, B \cdots \vdash ---}{\ldots, B, A \cdots \vdash ---}\ \textsc{Exchange-L} \qquad \frac{\ldots \vdash \cdots\cdots, A, B, \cdots\cdots}{\ldots \vdash \cdots\cdots, B, A, \cdots\cdots}\ \textsc{Exchange-R}$$

$$\frac{\ldots, A, A, \cdots \vdash ---}{\ldots, A, \cdots \vdash ---}\ \textsc{Contract-L} \qquad \frac{\ldots \vdash \cdots\cdots, A, A, \cdots\cdots}{\ldots \vdash \cdots\cdots, A, \cdots\cdots}\ \textsc{Contract-R}$$

**Cut Rule: Redundant but Convenient**

$$\frac{\ldots \vdash A, --- \qquad \ldots, A \vdash \boldsymbol{-\,-}}{\ldots \vdash ---, \boldsymbol{-\,-}}\ \textsc{Cut}$$

1. Prove $p \vee q, \neg p \vdash q$ using Sequent Calculus. (Hint: work bottom-up.)

2. Prove $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$. *First* use the sequent calculus. Then show a "block tableau" proof.

3. Prove $\neg(\neg p \wedge \neg q) \vdash p \vee q$ using sequent calculus.

4. The rules shown only cover propositional logic. Suggest some reasonable left and right rules for $\forall$ and $\exists$.

5. (In any reasonable logic) if we assume $A \vee B \vee C \vee D$ and $\neg A \vee E \vee F \vee G$, what can we conclude?

6. *Resolution* is another proof technique, historically used in theorem-proving applications. Parts of this idea live on in modern "SAT solvers," surprisingly practical tools for solving NP-complete problems, and in the way that Prolog works.

    - Again, all our proofs are by contradiction. We start with the assumption(s) and the negation of the conclusion, and try to derive a contradiction.
    - We first combine all formulas into "clausal form," a collection (implicitly a conjunction) of clauses, each of which is a disjunction of atomic formulas.
    - Repeatedly apply the resolution rule to overlapping pairs of clauses; discard tautologies (clauses that contain $A$ and $\neg A$).
    - The empty clause (disjunction of zero things) is equivalent to false, and hence gives us the contradiction we are searching for.

    - Prove, using resolution, that $((p \to q) \land (q \to r)) \to (p \to r)$.

    - Prove, using resolution, that $p \to q,\ \neg p \to r,\ \neg q \to \neg r \vdash q$

7. A unification of two terms is a substitution (replacing free *variables* by terms) that makes two formulas equal. Describe unifiers, if any, for:

   - $P(c)$ and $P(x)$, where $c$ is a constant.
   - $P(a, b, f(c))$ and $P(x, b, f(z))$
   - $P(x, x)$ and $P(f(y), f(z))$
   - $P(x, y)$ and $P(z, f(z))$
   - $P(x, f(x))$ and $P(g(y), w)$
   - $P(x, f(x))$ and $P(f(y), y)$
   - $P(x, y)$ and $P(f(y), g(z))$
   - $P(x, f(x))$ and $P(f(z), u)$
   - $P(f(x), g(x))$ and $P(f(u), u)$
   - $P(f(x), f(x))$ and $P(c, c)$
   - $P(f(x), g(x))$ and $P(y, g(y))$