CS81 **Assignment 4**
Due Wednesday, 16 February 2011

Prove sequents 1-2 by natural deduction.

1. [20 points] $(\forall x A(x)) \rightarrow \exists x B(x) \vdash \exists x(A(x) \rightarrow B(x))$

2. [15 points] $\exists x T \vdash \exists x(A(x) \rightarrow \forall y A(y))$

   (Here T is "top", so $\exists x T$ just means "something exists". Thus you can open a box with a fresh variable for use with the $\exists E$ rule.)

   For each of the entailments in 3-4 below, use interpretations (rather than a natural deduction proof) to argue that the entailment is true, or to give a counterexample:

3. [10 points] $\forall x(A(x) \vee B(x)) \models (\forall x A(x)) \vee (\exists x B(x))$

4. [10 points] $\forall x(A(x) \vee B(x)) \models (\forall x A(x)) \vee (\forall x B(x))$

5. [10 points] Prove the following using natural deduction with equality rules:

   $\forall x \forall y\ ((P(x, z) \wedge P(y, z)) \rightarrow x = y)$
   $\vdash (P(a, c) \wedge P(b, d) \wedge P(c, e) \wedge P(d, e)) \rightarrow a = b$

6. [15 points] Below are the four axioms for Abelian Groups:

   AG1: $\forall x\ \forall y\ \forall z\ ((x + y) + z = x + (y + z))$
   AG2: $\forall x\ (x + 0 = x)$
   AG3: $\forall x\ \exists y\ (x + y = 0)$
   AG4: $\forall x\ \forall y\ (x + y = y + x)$

   Give a natural deduction proof of the following cancellation law from those axioms:

$$\forall x\ \forall y\ \forall z\ (x + y = x + z) \rightarrow (y = z)$$

Above, + is a 2-ary function symbol, used in infix form, and 0 is a constant symbol. You may assume the following rules for equality, although they could be derived from the two natural deduction rules.

ER: $\forall x\ (x = x)$
ES: $\forall x\ \forall y\ ((x = y) \to (y = x))$
ET: $\forall x\ \forall y\ \forall z\ ((x = y) \wedge (y = z) \to (x = z))$
Sub+: $\forall u\ \forall v\ \forall x\ \forall y\ (((u = v) \wedge (x = y)) \to\ (u + x = v + y))$
Sub=: $\forall u\ \forall v\ \forall x\ \forall y\ (((u = v) \wedge (x = y)) \to\ ((u = x) \to (v = y)))$

7. [20 points] Below are axioms for the Natural Numbers:
Here ’ is a 1-ary function symbol denoting successor, written in postfix form, + is a 2-ary function symbol, 0 is a constant symbol, and = is equality.

N1: $\forall x\ \forall y\ (x' = y' \to x = y)$
N2: $\forall x\ \neg(x' = 0)$
N3: $\forall x\ (x + 0 = x)$
N4: $\forall x\ \forall y\ (x + y' = (x + y)')$
Induction: For any formula φ, and variable x:

$$(\varphi[0/x] \wedge \forall x\ (\varphi \to \varphi[x' / x])) \to \forall x\ \varphi$$

Note that Induction is an infinite set of axioms, one for each formula φ. We also have equality rules for +, ’, and = as in the previous problem.

Using natural deduction, prove the commutative law of addition:

$$\forall x\ \forall y\ (x + y) = (y + x)$$

You may use examples proved in the lecture as lemmas to simplify your proof. Consider first proving the special case with x = 0, which may involve a separate induction proof.

Sample Proof: **NT1: $\forall z\ \forall x\ \forall y\ (x + y) + z = x+ (y + z)$**

Here we identify φ with $\forall x\ \forall y\ (x + y) + z = (x+ y) + z$, so that Induction can be used. We will need to show $\varphi[0/z] \wedge \forall z\ (\varphi \rightarrow \varphi[z'/z])$ in order to get our conclusion $\forall z\varphi$ using the induction axiom instance in line 1.

1.	$((\forall x\ \forall y\ (x + y) + 0 = x+ (y + 0)) \wedge$ $(\forall z\ ((\forall x\ \forall y\ (x + y) + z = x+ (y + z)) \rightarrow (\forall x\ \forall y\ (x + y) + z' = x+ (y + z'))$ $\rightarrow \forall z\ \forall x\ \forall y\ (x + y) + z = x + (y + z)$	
2.	x_0	
3.	y_0	
4.	$y_0+0 = y_0$	N3, $\exists$E
5.	$(x_0+y_0)+0 = x_0+y_0$	N3, $\exists$E
6.	$(x_0+y_0)+0 = x_0+ (y_0+0)$	4, 5, =E
7.	$\forall y(x_0+y)+0 = x_0+ (y+0)$	3-6, $\forall$I
8.	$\forall x\forall y(x+y)+0 = x+ (y+0)$	2-7, $\forall$I
9.	z_0	
10.	$\forall x\ \forall y\ (x + y) + z_0 = x + (y + z_0)$	Assumption
11.	x_0	
12.	y_0	
13.	$(x_0 + y_0) + z_0 = x_0 + (y_0 + z_0)$	10, $\forall E^2$
14.	$((x_0 + y_0) + z_0)' = (x_0 + (y_0 + z_0))'$	13, sub '
15.	$(x_0 + y_0) + z_0' = ((x_0 + y_0) + z_0)'$	N4, $\forall E^2$
16.	$(x_0 + y_0) + z_0' = x_0 + (y_0 + z_0)'$	14, 15, =E
17.	$y_0 + z_0' = (y_0 + z_0)'$	N4, $\forall E^2$
18.	$(x_0 + y_0) + z_0' = x_0 + (y_0 + z_0')$	16, 17, =E
19.	$\forall y\ (x_0 + y) + z_0' = x_0 + (y + z_0)'$	12-18, $\forall$I
20.	$\forall x\ (x + y) + z_0' = x + (y + z_0)'$	11-19, $\forall$I
21.	$\forall x\ \forall y\ (x + y) + z_0 = x + (y + z_0))$ $\rightarrow (\forall x\ (x + y) + z_0' = x + (y + z_0)')$	10-20, $\rightarrow$I
22.	$\forall z\ ((\forall x\ \forall y\ (x + y) + z = x + (y + z))$ $\rightarrow (\forall x\ (x + y) + z' = x + (y + z)'))$	9-21, $\forall$I
23.	$(\forall x\forall y(x+y)+0 = x+ (y+0))$ $\wedge\ \forall z\ (((\forall x\ \forall y\ (x + y) + z = x + (y + z))$ $\rightarrow (\forall x\ (x + y) + z' = x + (y + z)')))$	8, 22, $\wedge$I
24.	$\forall x\ \forall y\ (x + y) + z = x + (y + z)$	1, 23, $\rightarrow$E

Notes:

sub ’ means the equality substitution applied to the ’ function:

$s = t \rightarrow s' = t'$ for any terms s, t

=E means equality elimination

$\forall E^2$ means ∀E applied twice in succession

NT2: $\forall y\ \forall x\ (x' + y) = x + y'$

Proof by induction on y.

1.	$((\forall x\ (x' + 0) = x + 0') \wedge$ $\forall y\ ((\forall x\ (x' + y) = x + y') \rightarrow \forall x\ (x' + y') = x + (y')')\ ) \rightarrow$ $\forall y\ \forall x\ (x' + y) = x + y'$	Induction axiom
2.	x_0	
3.	$x_0' + 0 = x_0'$	N3, ∀E
4.	$x_0 + 0 = x_0$	N3, ∀E
5.	$x_0 + 0' = x_0'$	4, Sub ’
6.	$x_0' + 0 = x_0 + 0'$	2, 5, =E
7.	$(\forall x\ (x' + 0) = x + 0')$	2-6, ∀I
8.	y_0	
9.	$\forall x\ (x' + y_0) = x + y_0'$	Assumption
10.	x_0	
11.	$x_0' + y_0 = x_0 + y_0'$	9, ∀E
12.	$x_0' + y_0' = (x_0' + y_0)'$	N4, $\forall E^2$
13.	$x_0' + y_0' = (x_0 + y_0')'$	11, 12, =E
14.	$x_0 + (y_0')' = (x_0 + y_0')'$	N4, $\forall E^2$
15.	$x_0' + y_0' = x_0 + (y_0')'$	13, 14, =E
16.	$\forall x\ (x' + y_0' = x + (y_0')')$	10-15, ∀I
17.	$(\forall x\ (x' + y_0) = x + y_0') \rightarrow (\forall x\ (x' + y_0' = x + (y_0')'))$	9-16, →I
18.	$\forall y\ ((\forall x\ (x' + y) = x + y') \rightarrow (\forall x\ (x' + y' = x + (y')')))$	8-17, ∀I
19.	$(\forall x\ (x' + 0) = x + 0') \wedge$ $\forall y\ ((\forall x\ (x' + y) = x + y') \rightarrow (\forall x\ (x' + y' = x + (y')')))$	7, 18, ∧I
20.	$\forall y\ \forall x\ (x' + y) = x + y'$	1, 19, →E