Assignment 11: (Un)decidability

Due: Wednesday, April 27

- Emails about this assignment should be directed to `cs81help@cs.hmc.edu`.
- Grutor office hours in Platt are Sundays 8–10pm, Mondays 7–11pm, and Tuesdays 7pm-12m. Prof. Stone's official office hours in Olin 1251 are MW 4–5pm and TR 3–4pm, and Prof. Keller's official office hours in Olin 1253 are MTW 4-5:30pm; you can often catch us at other times (or make an appointment).
- Make sure your submission includes your name!

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1. Read Chapter 4 and Section 5.1 of Sipser. Come up with (at least) two questions about the reading where you're not sure of the answer. These may relate to points where the book is confusing, or simply to some related question or conjecture that occurs to you while doing the reading.

2. Consider the following languages (where M ranges over TMs). Are they recognizable? Are they decidable? Prove each answer.

$$\begin{aligned}
L_1 &= \{ \langle M \rangle \mid M \text{ accepts at least 999 strings} \} \\
L_2 &= \{ \langle M, w \rangle \mid M \text{ moves right exactly twice when running on input } w \} \\
L_3 &= \{ \langle M \rangle \mid M\text{'s program includes a transition that writes } \mathtt{a} \text{ on the tape} \} \\
L_4 &= \{ \langle M \rangle \mid M \text{ eventually writes a non-blank symbol to the tape when given input } \varepsilon \} \\
L_5 &= \{ \langle M \rangle \mid M \text{ accepts the string } a \} \\
L_6 &= \{ \langle M \rangle \mid M \text{ is the only TM accepting } L(M) \} \\
L_7 &= \{ \langle M \rangle \mid M \text{ halts on } \epsilon \text{ in no more than 1000 steps} \}
\end{aligned}$$

3. We know that A_{TM} is recognizable but not decidable, and its complement $\neg A_{TM}$ is neither recognizable nor decidable. Is the third language

$$R_{TM} = \{ \langle M, w \rangle \mid M \text{ rejects } w \}$$

decidable, recognizable, or neither? Prove your answer.

4. In a previous homework, you showed that the language

$$\mathit{ADD} = \{ x{=}y{+}z \mid x, y, z \text{ are binary integers, and } x \text{ is the sum of } y \text{ and } z \}$$

was not regular. You can surely write a program (in Python, or Java, or Turing-Machine, etc.) that takes a string and decides whether it's one of these correct sums of binary numbers; *ADD* is clearly a decidable language.[1]

[1] As we discussed a few weeks ago, if *ADD* is decidable then addition is computable; if you want to add 3 and 5, consider all binary number s in increasing order, check whether "$s = 11 + 101$" is in *ADD*, and stop when you find an s where the answer is yes.

Now, even if you restrict yourself to a single programming language, there might be many programs to decide *ADD*; they each decide whether the input is a correct binary sum or now, each in slightly different (or very different) ways. We can then consider the set of all such programs.

(a) Prove that the language

$$\mathit{ADDERS} = \{\, \langle M \rangle \mid L(M) = \mathit{ADD} \,\}$$

is not decidable.

(b) Prove that *ADDERS* is also not recognizable by showing $\neg A_{TM} \leqslant \mathit{ADDERS}$. That is, reduce $\neg A_{TM}$ to *ADDERS* by showing that if *ADDERS* were *recognizable*, then $\neg A_{TM}$ would be recognizable too. (You may assume that *ADD* is decidable, i.e., there is at least one Turing Machine that decides it.)

(c) If we assume our encoding of Turing Machines is such that every string represents a Turing Machine, then the complement of *ADDERS* is the set

$$\neg\mathit{ADDERS} = \{\, \langle M \rangle \mid L(M) \neq \mathit{ADD} \,\}$$

Is $\neg$*ADDERS* decidable? Is it recognizable? Prove your answers.