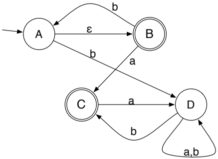
Introduction to Computability

March 21, 2011

CS 81: Computability and Logic

Selected Topics in Computability

Finite State Machines



Regular Expressions

a(a|b)*a

Context-Free Grammars

E → n
E → E + n

Turing Machines

Partial Recursive Functions

h(μx.[f(g(x))=0])

Lambda Calculus & Combinatory Logic (CS 131)

λf.λb.f(f(b))

S(SII)(SII)KS

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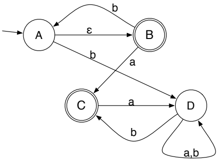

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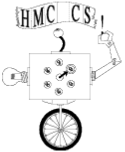

Partial Recursive Functions

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If these are the answers, what is the question?

Common Simplifying Assumptions

1. Encoding inputs as *finite strings.*
   - Generalizing "everything's just bits"

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1. Encoding inputs as *finite strings.*
   - Generalizing "everything's just bits"
2. A focus on *decision problems*
   - Described as the set of inputs (strings) where the answer is "yes"

Alphabet

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The elements of Σ are called *letters* or *symbols*.

- ✓ We will always assume a finite alphabet.
- ✓ Examples?

Strings

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$$c_1 c_2 \cdots c_n$$

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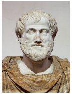

What about my alphabet?

Languages

A *language* L *over* Σ is a set of strings over Σ.

Languages

> A *language* L over Σ is a set of strings over Σ.

- ✓ The set of all strings over Σ is written Σ^*.
- ✓ The empty set $\emptyset$ is a language.
- ✓ Languages may be finite or infinite, but they contain only finite strings!
- ✓ Other examples?

Strings (more formally)

The set Σ^* can be defined inductively:

- ✓ $\varepsilon \in \Sigma^*$
- ✓ If $a \in \Sigma$ and $x \in \Sigma^*$ then $a \bullet x \in \Sigma^*$

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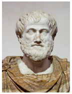
✓ 

Am I in Σ^* ?

Resulting Induction Principles

- ✓ **Structural induction on strings:** If $P(\varepsilon)$ and

$$\forall x \in \Sigma^*. \forall a \in \Sigma. \quad P(x) \to P(ax)$$

then $\forall w \in \Sigma^*. P(w)$.

Resulting Induction Principles

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then $\forall w \in \Sigma^*.\ P(w)$.

- ✓ **Induction by length:** If $P(\varepsilon)$ and

$$\begin{aligned}\forall n > 0. \quad & (\forall w \in \Sigma^*. \mathtt{length}(w) = n{-}1 \to P(w)) \\ & \to (\forall w \in \Sigma^*. \mathtt{length}(w) = n \to P(w))\end{aligned}$$

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✓ **Strong induction by length:** If

$$\begin{aligned} \forall n \geqslant 0. \quad & (\forall w \in \Sigma^*. \mathtt{length}(w) < n \to P(w)) \\ & \to (\forall w \in \Sigma^*. \mathtt{length}(w) = n \to P(w)) \end{aligned}$$

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  then $\forall w \in \Sigma^*. P(w)$.

The inductive definition of strings also justifies defining *functions* by induction/recursion.

Operations on Strings

```
append(ε,y)  := y
append(ax,y) := a●append(x,y)

length(ε)  := 0
length(ax) := 1 + length(x)

rev(ε)  := ε
rev(ax) := append(rev(x),a)
```

Operations on Strings

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append(ε,y)  := y                  εy    := y
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length(ε)  := 0                    |ε|  := 0
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rev(ε)  := ε                       ε^R     := ε
rev(ax) := append(rev(x),a)        (ax)^R  := x^R a
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```

Prove

- ✓ $\forall w \in \Sigma^*. (w\varepsilon = w)$
- ✓ $\forall w \in \Sigma^*. \forall y, z \in \Sigma^*. (wy)z = w(yz)$
- ✓ $\forall w \in \Sigma^*. \forall z \in \Sigma^*. \mathtt{length}(wz) = \mathtt{length}(w) + \mathtt{length}(z)$
- ✓ $\forall w \in \Sigma^*. \forall z \in \Sigma^*. (wz)^R = z^R w^R$

Operations on Languages

$$L \cup M$$

$$L \cap M$$

$$L \setminus M$$

$$LM := \{\, xy \mid x \in L,\ y \in M \,\}$$

$$L^n \qquad L^0 := \{\varepsilon\}, \quad L^{n+1} := LL^n$$

$$L^* := L^0 \cup L^1 \cup L^2 \cup \cdots$$

$$L^+ := L^1 \cup L^2 \cup \cdots$$

Example

Assume $L = \{ \mathsf{ba}, \mathsf{da} \}$ and $M = \{ \mathsf{da}, \mathsf{rk}, \varepsilon \}$.

$$
\begin{aligned}
L \cup M &= \\
L \cap M &= \\
L \setminus M &= \\
LM &= \\
L^3 &= \\
L^* &= \\
L^+ &=
\end{aligned}
$$

Language Equivalences (not the same L)

- ✓ $L\emptyset =$
- ✓ $L\{\varepsilon\} =$
- ✓ $\{\varepsilon\}^* =$
- ✓ $\{\varepsilon\}^+ =$
- ✓ $\emptyset^* =$
- ✓ $\emptyset^+ =$
- ✓ $(L \cup M)N =$
- ✓ $(L^*)^* =$