Regular Languages, Continued Again

March 30, 2011
CS 81: Computability and Logic

Closure Properties

A *family* of languages is a set of languages.

- ✓ The family of all finite languages
- ✓ The family of all languages
- ✓ The family of all regular languages

A family $\mathcal{F}$ is *closed under* an operation if applying the operation to languages in $\mathcal{F}$ always produces a result in $\mathcal{F}$.

Finite Languages

Is the family of finite languages closed under:

- ✓ Union? $(A \cup B)$
- ✓ Intersection $(A \cap B)$
- ✓ Concatenation? (AB)
- ✓ Star (A^*)
- ✓ Complement (A^c)

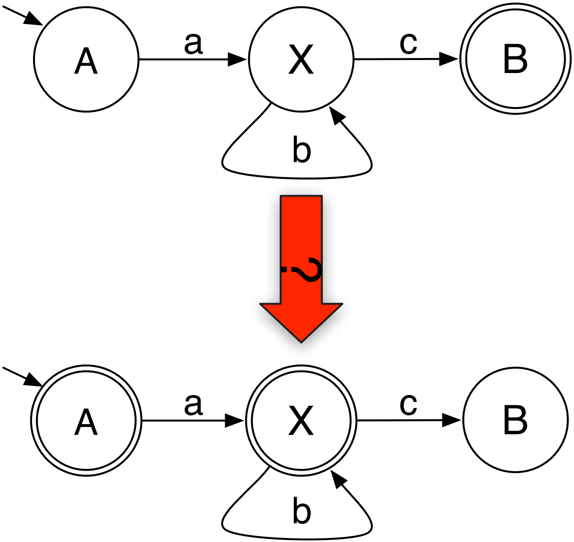
Regular Languages

The regular languages are closed under

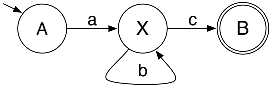
- ✓ Union? $(A \cup B)$
- ✓ Intersection $(A \cap B)$
- ✓ Concatenation? (AB)
- ✓ Star (A^*)
- ✓ Complement (A^c)

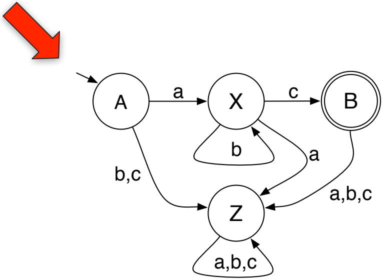
Proofs: Consider the corresponding automata...

Complement?



COMPLEMENT!





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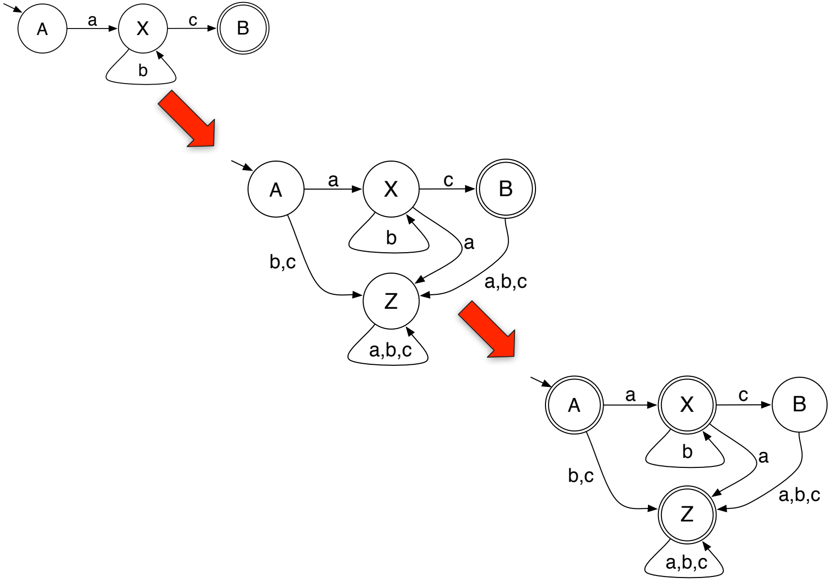

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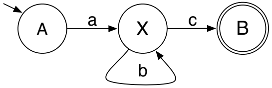
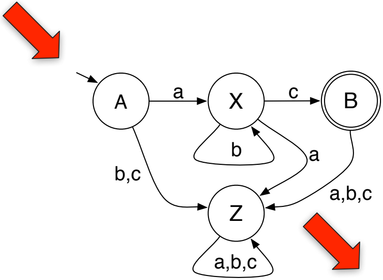
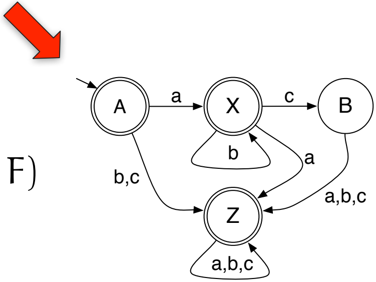
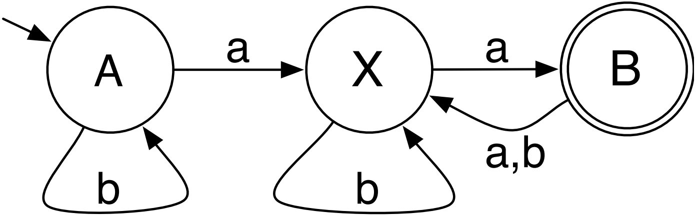


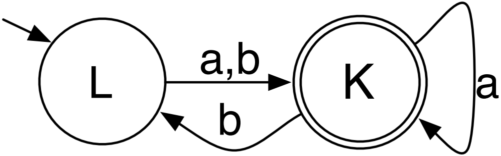




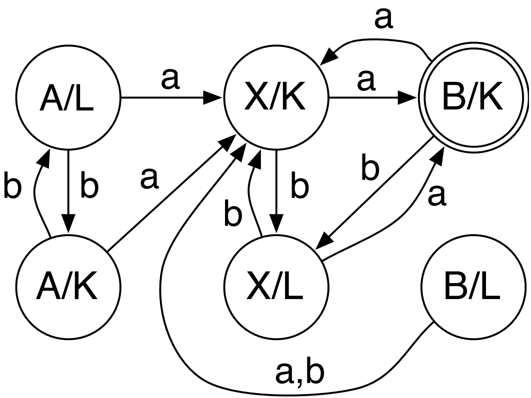
$$\text{DFA } M = (\Sigma, Q, \rightarrow, q_0, F)$$
$$\text{DFA } M^c = (\Sigma, Q, \rightarrow, q_0, Q \setminus F)$$

Intersection: DFA Inputs





Intersection: Product Automaton



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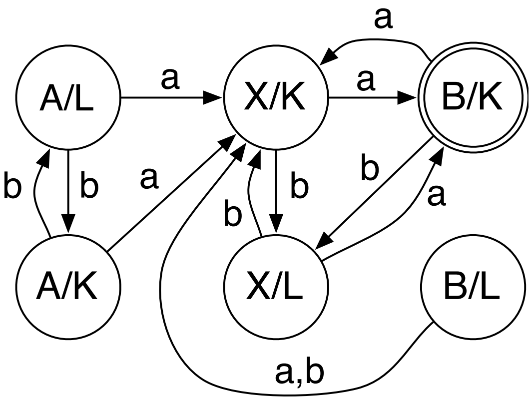
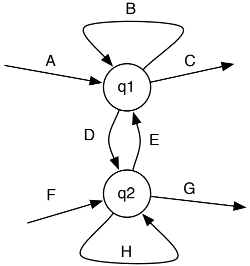


$$\text{DFA } M = (\Sigma, Q, \rightarrow, q_0, F)$$
$$\text{DFA } M' = (\Sigma, Q', \rightarrow', q_0', F')$$
$$\text{DFA } M \cap M' = (\Sigma, Q \times Q', \rightarrow_{\text{both}}, \langle q_0, q_0' \rangle, F \times F').$$

State Machine Optimization

If two states have the same language, they can be merged without changing the language of the state machine.



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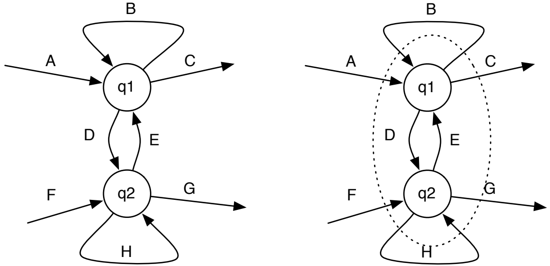

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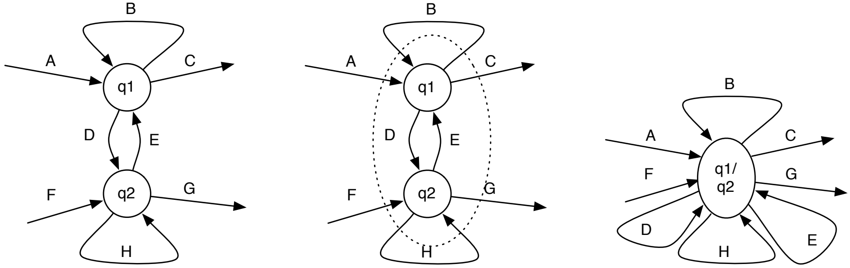
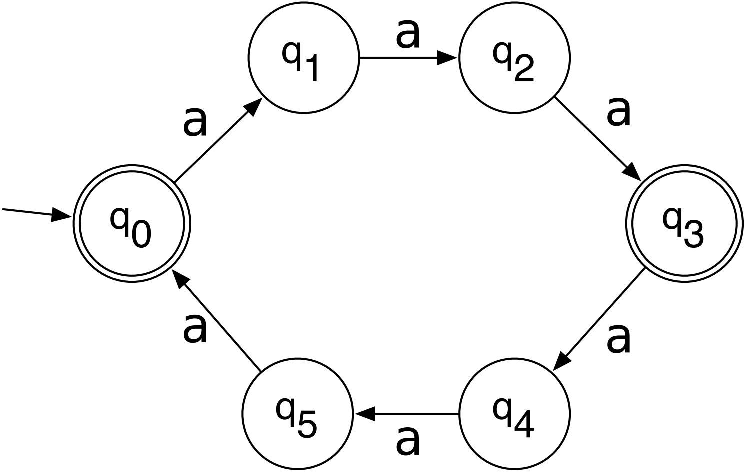


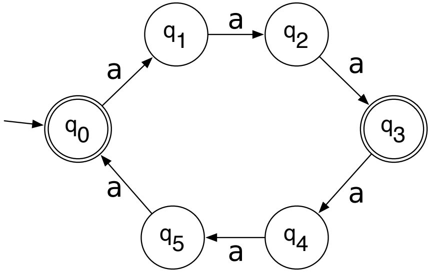
Example



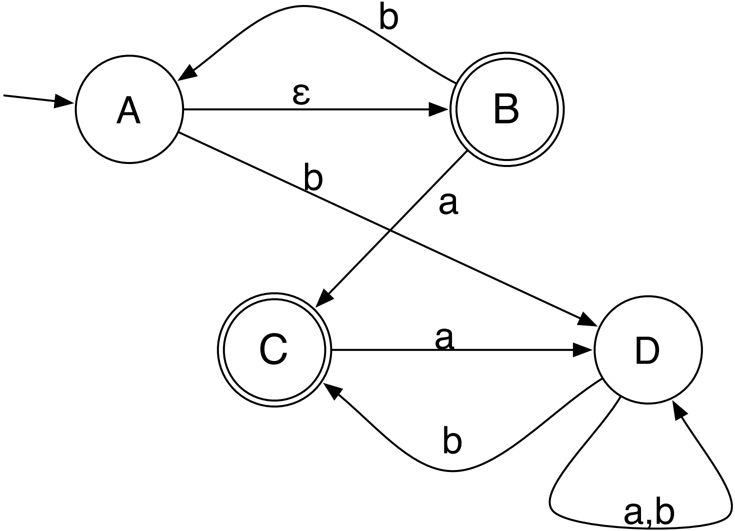
One DFA Minimization Algorithm

Assume all states are mergable unless there's evidence otherwise:

- ✓ Accepting vs. nonaccepting.
- ✓ Same-symbol transitions to known-different states.



More Complex Example



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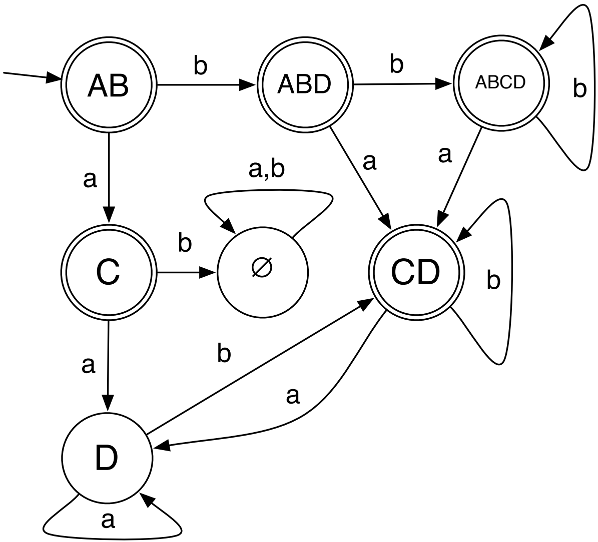

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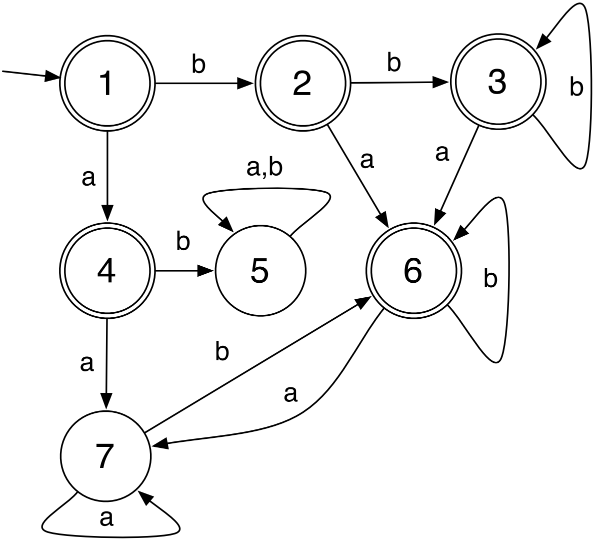

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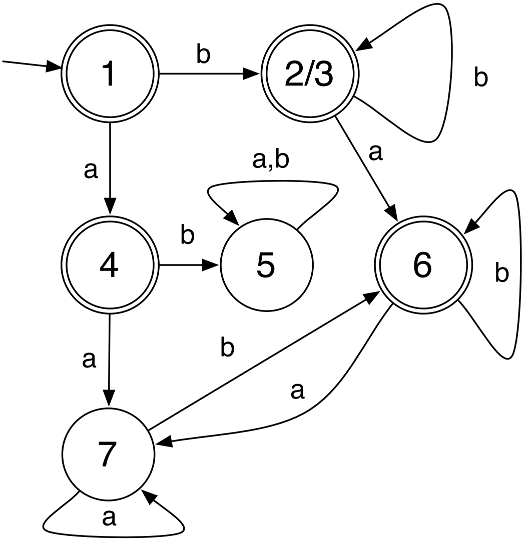

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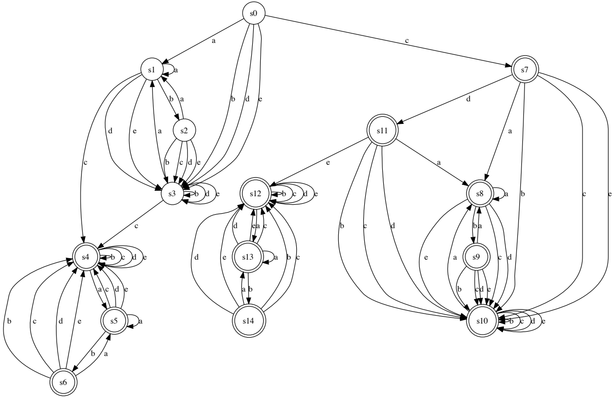

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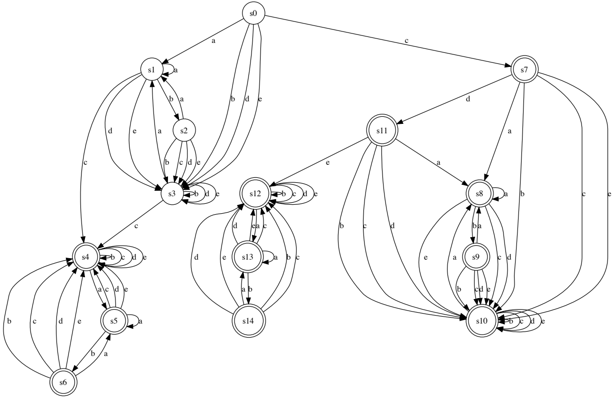

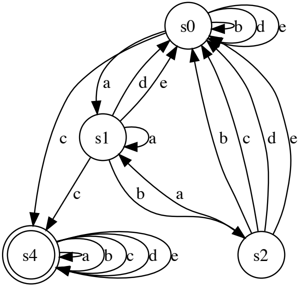

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For any language L and $x \in \Sigma^*$, *define*

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In a minimal DFA, we would have exactly one state whose language is $\partial_x L$.

Consequence

Theorem (Myhill-Nerode (essentially))

A language is regular iff $\{ \partial_x L \mid x \in \Sigma^* \}$ *is finite.*

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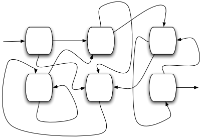
$$\partial_\varepsilon L \neq \partial_0 L \neq \partial_{00} L \neq \partial_{000} L \neq \cdots$$

Maze Theory

- ✓ Suppose you enter a maze of twisty passages, all alike. (but with doors that open from only one side)



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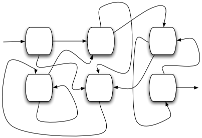

- ✓ You happen to know that your maze has exactly 19 rooms.
You start wandering and pass through 27 rooms.
What can you conclude?

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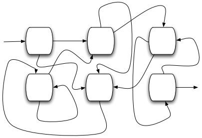

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What can you conclude about other solutions to the maze?

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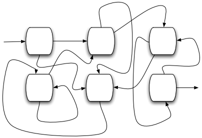

- You happen to know that your maze has exactly 19 rooms. You start wandering and pass through 27 rooms. What can you conclude?

- This wandering has brought you to an exit. What can you conclude about other solutions to the maze?

- Was there anything special about the numbers 19 and 27?

Finite Maze Theorem

For every finite maze there is a number p, such that

For every path through the maze s with $|s| \geqslant p$:

- The path s contains at least one loop, which starts and ends within the first p steps.
- There are infinitely many paths through the maze(at least one shorter, and arbitrarily many longer) whose lengths differ by a multiple of some constant.

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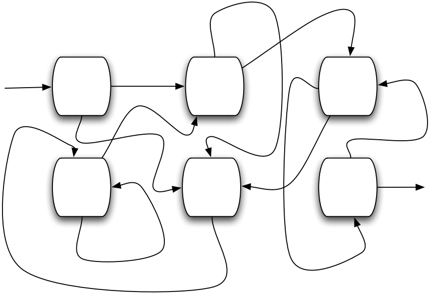
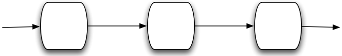
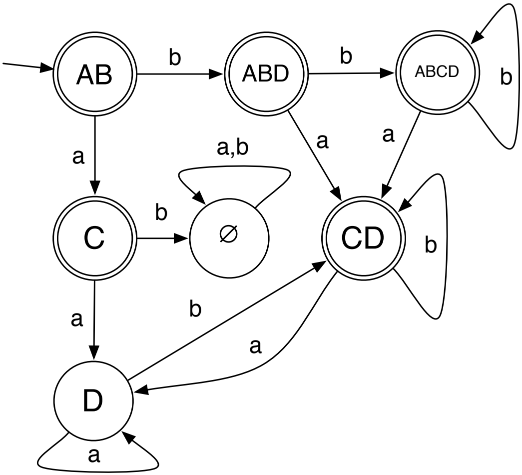




Finite Automata As Mazes



A Pumping Lemma

If L is a regular language, then
there exists a number p such that
For every $s \in L$ with $|s| \geqslant p$
we can decompose s into xyz where

1. $y \neq \varepsilon$
2. $|xy| \leqslant p$
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What logically follows?

- ✓ If every long string in L can be pumped, then L is regular
- ✓ If there's a long string in L that can be pumped, then L is regular
- ✓ If not every long string in L can be pumped, then L isn't regular
- ✓ If there's a long string in L that can't be pumped, then L isn't regular

Deriving a Useful Corollary

The Pumping Lemma tells us that:

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What logically follows?

- ✓ If not every long string in L can be pumped, then L isn't regular!
- ✓ If there's a long string in L that can't be pumped, then L isn't regular!

Using the Pumping Lemma

To prove a language *isn't* regular:

- ✓ Suppose L were regular, with pumping length p
  - Carefully pick a long ($\geqslant p$) string $s \in L$
  - Show that s cannot be pumped
- ✓ Contradiction. Therefore, L is not regular.

You *cannot* use it to prove a language is regular!

- ✓ E.g., non-regular languages with *every* string pumpable

$$\{a^i b^j c^j \mid i \geqslant 1, j \geqslant 0\} \cup \{b^j c^k \mid j, k \geqslant 0\} \qquad p = 1$$

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So L is not regular. QED.