Regular Languages, Continued Again

March 30, 2011

CS 81: Computability and Logic
Closure Properties

A family of languages is a set of languages.

✓ The family of all finite languages
✓ The family of all languages
✓ The family of all regular languages

A family $F$ is closed under an operation if applying the operation to languages in $F$ always produces a result in $F$. 
Finite Languages

Is the family of finite languages closed under:

✓ Union? \((A \cup B)\)
✓ Intersection \((A \cap B)\)
✓ Concatenation? \((AB)\)
✓ Star \((A^*)\)
✓ Complement \((A^c)\)
Regular Languages

The regular languages are closed under

✓ Union? \((A \cup B)\)
✓ Intersection \((A \cap B)\)
✓ Concatenation? \((AB)\)
✓ Star \((A^*)\)
✓ Complement \((A^c)\)

Proofs: Consider the corresponding automata...
COMPLEMENT?
**COMPLEMENT!**

Graph representation of a complement of a given automaton.
COMPLEMENT!

\[
\begin{align*}
\text{X} & \text{A} \\
& a \quad \text{c} \\
\text{B} & \text{b} \\
\end{align*}
\]

\[
\begin{align*}
\text{A} & \text{X} \\
& a \quad \text{c} \\
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COMPLEMENT!

\[ \text{DFA } M = (\Sigma, Q, \rightarrow, q_0, F) \]
\[ \text{DFA } M^c = (\Sigma, Q, \rightarrow, q_0, Q \setminus F) \]
INTERSECTION: DFA INPUTS
INTERSECTION: PRODUCT AUTOMATON
**Intersection: Product Automaton**

DFA $M = (\Sigma, Q, \rightarrow, q_0, F)$

DFA $M' = (\Sigma, Q', \rightarrow', q'_0, F')$

DFA $M \cap M' = (\Sigma, Q \times Q', \rightarrow_{both}, \langle q_0, q'_0 \rangle, F \times F')$. 
**State Machine Optimization**

If two states have the same language, they can be merged without changing the language of the state machine.
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EXAMPLE
One DFA Minimization Algorithm

Assume all states are mergable unless there’s evidence otherwise:

✓ Accepting vs. nonaccepting.
✓ Same-symbol transitions to known-different states.
**More Complex Example**

A graph with states A, B, C, and D, labeled with transitions labeled with symbols a, b, and ε, illustrating the concept of a more complex example in formal language theory.
**More Complex Example**

```
A B b

C a, b

D a, b

A B a

A B D b

A B C D a

B C b

C D a, b

C D b

D a

A B C D a
```
More Complex Example
More Complex Example

![Diagram of a more complex example in automata theory](image-url)
Bigger Example
**Bigger Example**
**Derivatives of a Language**

For any language $L$ and $x \in \Sigma^*$, define

$$\partial_x L := \{ y \in \Sigma^* \mid xy \in L \}$$
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So, if you run $x$ through a state machine for $L$, you end up in a state whose language is $\partial_x L$.

In a minimal DFA, we would have exactly one state whose language is $\partial_x L$. 
**Consequence**

Theorem (Myhill-Nerode (essentially))

A language is regular iff \( \partial_x L \mid x \in \Sigma^* \) is finite.
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*A language is regular iff* \( \partial_x L | x \in \Sigma^* \) *is finite.*

Proof idea: The size of this set is the size of the smallest deterministic finite-state machine.
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Consider

\( L := \{ a^{3n} \mid n \geq 0 \} \)
Theorem (Myhill-Nerode (essentially))

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$L := \{ a^{3n} \mid n \geq 0 \}$

$\partial_\varepsilon L = \partial_{aaa} L = \cdots$, $\partial_a L = \partial_{aaa} a L = \cdots$, $\partial_{aa} L = \partial_{aaa} a a L = \cdots$
Consequence

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\( \partial_\varepsilon L = \partial_{aaa} L = \cdots \), \( \partial_a L = \partial_{aaa} L = \cdots \), \( \partial_{aa} L = \partial_{aaa} L = \cdots \)

✓ \( L := \{ 0^n 1^n \mid n \geq 0 \} \)
Consequence

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\[ L := \{ 0^n 1^n \mid n \geq 0 \} \]
\[ \partial_\varepsilon L \neq \partial_0 L \neq \partial_{00} L \neq \partial_{000} L \neq \cdots \]
Maze Theory

✓ Suppose you enter a maze of twisty passages, all alike. (but with doors that open from only one side)
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✓ You happen to know that your maze has exactly 19 rooms. You start wandering and pass through 27 rooms. What can you conclude?
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✓ This wandering has brought you to an exit. What can you conclude about other solutions to the maze?
Maze Theory

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✓ You happen to know that your maze has exactly 19 rooms. You start wandering and pass through 27 rooms. What can you conclude?

✓ This wandering has brought you to an exit. What can you conclude about other solutions to the maze?

✓ Was there anything special about the numbers 19 and 27?
Finite Maze Theorem

For every finite maze there is a number $p$, such that

For every path through the maze $s$ with $|s| \geq p$:

- The path $s$ contains at least one loop, which starts and ends within the first $p$ steps.
- There are infinitely many paths through the maze (at least one shorter, and arbitrarily many longer) whose lengths differ by a multiple of some constant.
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FINITE AUTOMATA AS MAZES

Diagram of a finite automaton with states labeled as follows:
- AB
- ABD
- ABCD
- C
- CD
- D

Transitions:
- AB to ABD on b
- ABD to ABCD on b
- C to D on a
- D to ABD on a
- ABD to D on a
- ABCD to C on b
- CD to ABD on a
- CD to D on b
- D to ABCD on a
A PUMPING LEMMA

If $L$ is a regular language, then

there exists a number $p$ such that

For every $s \in L$ with $|s| \geq p$

we can decompose $s$ into $xyz$ where

1. $y \neq \varepsilon$
2. $|xy| \leq p$
3. $xy^iz \in L$ for every $i \geq 0$. 
A Pumping Lemma

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Deriving a Useful Corollary

The Pumping Lemma tells us that:

If $L$ is regular,
then every long-enough string in $L$ can be pumped.
**Deriving a Useful Corollary**

The Pumping Lemma tells us that:

If $L$ is regular,

then every long-enough string in $L$ can be pumped.

What logically follows?

✓ If every long string in $L$ can be pumped, then $L$ is regular
✓ If there’s a long string in $L$ that can be pumped, then $L$ is regular
✓ If not every long string in $L$ can be pumped, then $L$ isn’t regular
✓ If there’s a long string in $L$ that can’t be pumped, then $L$ isn’t regular
Deriving a Useful Corollary

The Pumping Lemma tells us that:

If L is regular,
then every long-enough string in L can be pumped.

What logically follows?

✓ If not every long string in L can be pumped,
then L isn’t regular!

✓ If there’s a long string in L that can’t be pumped,
then L isn’t regular!
**Using the Pumping Lemma**

To prove a language isn’t regular:

- **✓** Suppose $L$ were regular, with pumping length $p$
  - Carefully pick a long ($\geq p$) string $s \in L$
  - Show that $s$ cannot be pumped

- **✓** Contradiction. Therefore, $L$ is not regular.

You cannot use it to prove a language is regular!

- **✓** E.g., non-regular languages with every string pumpable
  
  $$\{a^i b^j c^j | i \geq 1, j \geq 0\} \cup \{b^j c^k | j, k \geq 0\} \quad p = 1$$
\[ L = \{0^n1^n \mid n \geq 0\} \]

Suppose \( L \) were regular

✓ Let \( p \) be the pumping length
\[ L = \{ 0^n1^n \mid n \geq 0 \} \]

Suppose \( L \) were regular

- Let \( p \) be the pumping length
- Consider, for example, \( s := 0^p1^p \). (Note that \( |s| \geq p \).)
L = \{ 0^n1^n | n \geq 0 \}

Suppose L were regular

✓ Let p be the pumping length
✓ Consider, for example, s := 0^p1^p. (Note that |s| \geq p.)
✓ Consider all possible decompositions

\[ s = xyz \text{ with } y \neq \varepsilon \land |xy| \leq p. \]
\[ L = \{0^n1^n \mid n \geq 0\} \]

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- None of them work for pumping.
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Suppose $L$ were regular

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✓ Consider, for example, $s := 0^p1^p$. (Note that $|s| \geq p$.)

✓ Consider all possible decompositions

$$s = xyz \text{ with } y \neq \varepsilon \land |xy| \leq p.$$  

✓ None of them work for pumping.

✓ Contradiction.
\[ L = \{ 0^n 1^n \mid n \geq 0 \} \]

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✓ Let \( p \) be the pumping length
✓ Consider, for example, \( s := 0^p 1^p \). (Note that \( |s| \geq p \).)
✓ Consider all possible decompositions

\[ s = xyz \quad \text{with} \quad y \neq \varepsilon \land |xy| \leq p. \]

✓ None of them work for pumping.
✓ Contradiction.

So \( L \) is not regular. QED.