Computation Histories and PCP

(not Probabilistically Checkable Proofs)
(also not Angel Dust)

CS 81: Computability and Logic

April 20, 2011

Computation History

- Step-by-step recording of a TM computation.
- Used to show more problems not decidable.

States as Strings

- We can describe the configuration of any TM using a string $C = xqy \in \Gamma^*$.
  - $x \in \Gamma^*$ = symbols to the left of head
  - $q \in Q$ = current control state
  - $y \in \Gamma^*$ = symbols under and to the right of head
- Over the course of a computation, we have
  - $\cdots \Rightarrow x_1q_1y_1 \Rightarrow x_2q_2y_2 \Rightarrow x_3q_3y_3 \Rightarrow \cdots$
  - If the TM halts, we can represent the whole history of the computation as a single (finite) string!
  - Traditionally written $\#C_1\#C_2\#C_3\#\cdots\#C_n\#$

Checking a History

- Checker: a Turing Machine C that, given <M, h>, checks whether h is a history of TM M.
  - Consecutive states should be equal, except around the head (where the change corresponds to the transition table of M).
- Can check whether h is a halting (or an accepting) history by looking at the last control state.

Digression: LBAs

- In fact, the CH can be checked for validity by a less-than-general TM called an LBA.
  - LBA = "Linear Bounded Automaton," a TM that can only use the part of the tape containing input.
  - An LBA can have a large tape alphabet and can "mark" tape cells. It just can't grow its tape.
- More powerful than a DFA
  - Number of potential states grows with input size
  - DFA wouldn't be able to check a computation history.

Accepting for LBAs

- A_{LBA} = { <M,w> | M a LBA accepting w } is decidable.
- Proof: When running M on w, there are at most

$$n := |Q| \times |\Gamma|^{|w|} \times |w|$$

distinct “states” during the computation.

- So,
  - Run M on w.
  - If computation takes longer than n steps, its in an infinite loop; M doesn’t accept w.

Emptiness for LBAs

- E_{LBA} = { <M> | M a LBA, L(M) = ∅ } isn't decidable.
- Proof: $A_{TM} \leq E_{LBA}$.

All_{CFG}

All_{CFG} = { <G> | G a CFG, L(G) = Σ^* } is undecidable.

Proof: $A_{TM} \leq All_{CFG}$.

- Key: given <M,w> create a PDA/CFG for strings that **aren**'t accepting computation histories!
  - PDA accepts strings that
    - Don't start with q_0w
    - Or, don't end with $xq_{accept}y$
    - Or, two successive configurations don't match properly
  - Hack: need to reverse every other configuration.
- The grammar for this PDA is Σ^* iff M,w has no finite, accepting history.

Eq_{CFG}

$Eq_{CFG} = \{ \langle G_1, G_2 \rangle \mid L(G_1) = L(G_2) \}$ is undecidable.

Proof: $All_{CFG} \leq Eq_{CFG}$.

Post Correspondence Problem

Emil Post

- Named after logician Emil Post (1897-1954)
  - studied fundamental models of computation
  - “scooped” by Gödel, Turing, and Church

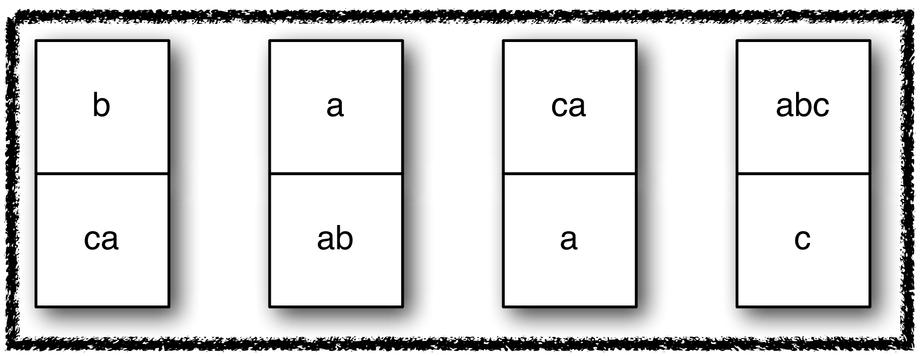
Why PCP?

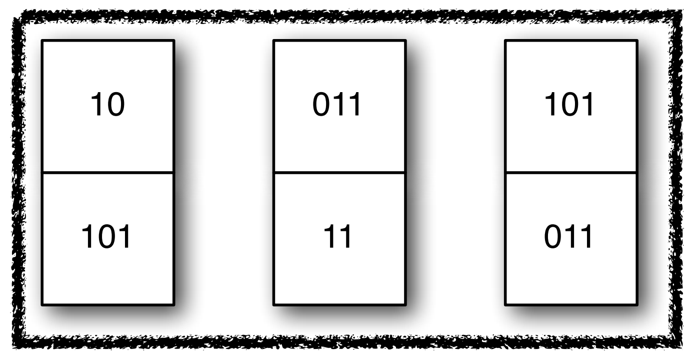
- Trivial problem to state
  - Looks nothing like Turing Machines
  - A child can understand it
  - Superficially, doesn't look that hard

- Can reduce PCP to other problems, showing them undecidable

PCP

- Given a set of “dominos” (pairs of strings), find a finite sequence of dominos, repeats allowed, where the top line and bottom line spell out the same string.

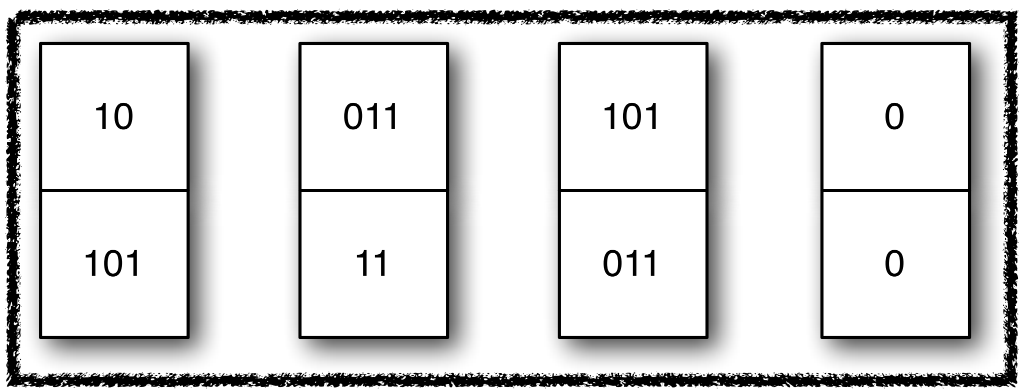




- Theorem: PCP is undecidable.
- Proof: Halting ≤ MPCP ≤ PCP

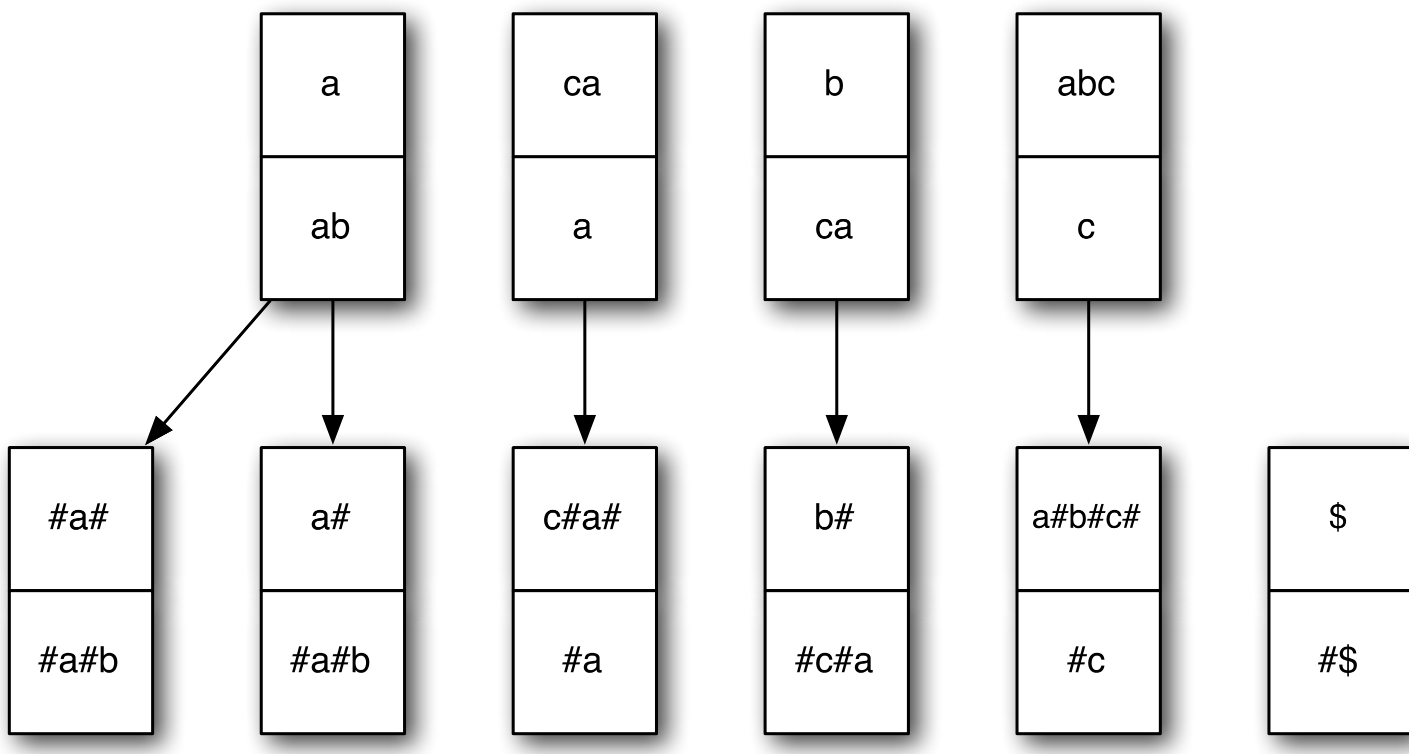
Modified PCP (MPCP)

- Like PCP, but solution starts with first domino.
- The following MPCP instance has no solution.



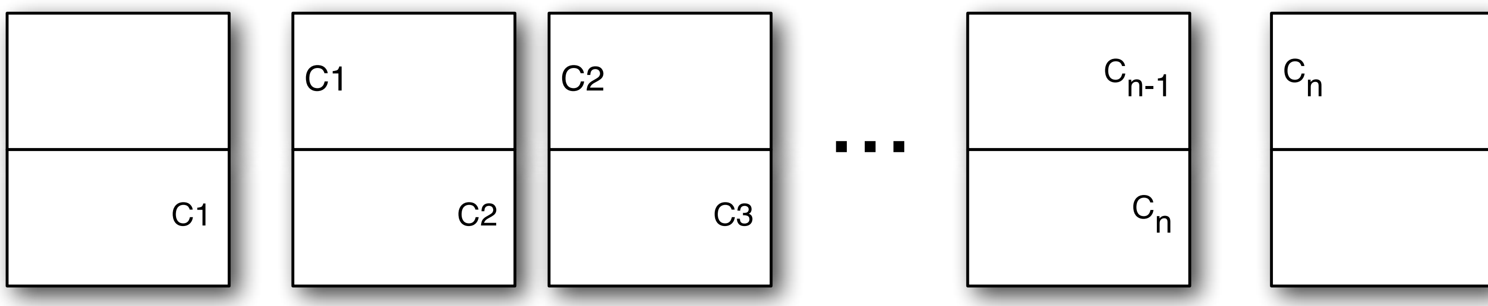
MPCP ≤ PCP

- Given an instance of MPCP, solve by translating dominos and doing PCP.



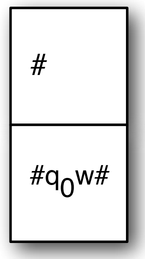
Halts ≤ MPCP

- Idea: if a TM halts, it has a computation history
- Given a <M,w>, construct dominos whose solution would yield such a history (on top and bottom); use MPCP-solver to check for a solution.

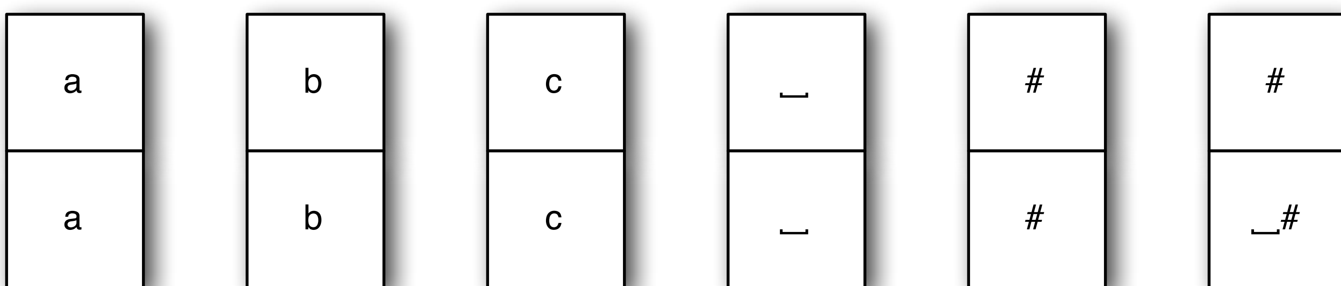


The Dominos

- First domino: set up the initial state

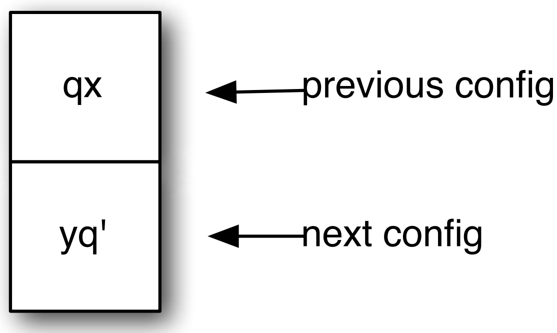


- Helper dominos: copy unchanged parts of the tape from C_i to C_{i+1}. (optional: expand the tape)

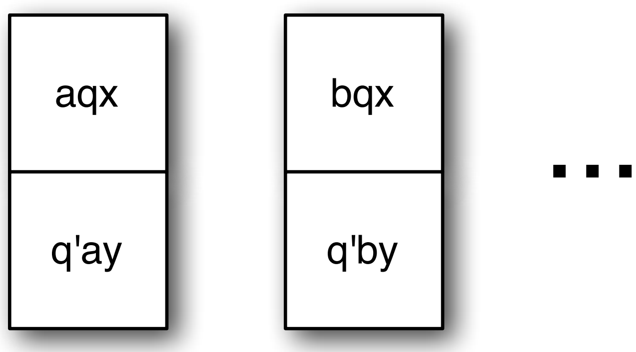


More Dominos

- For each transition $q,x \rightarrow q', y, R$

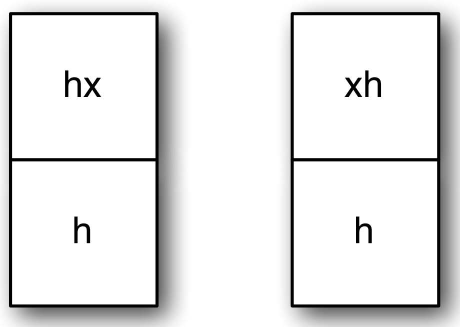


- For each transition $q,x \rightarrow q', y, L$

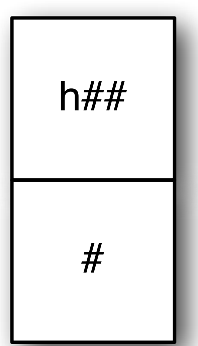


Completion Dominos

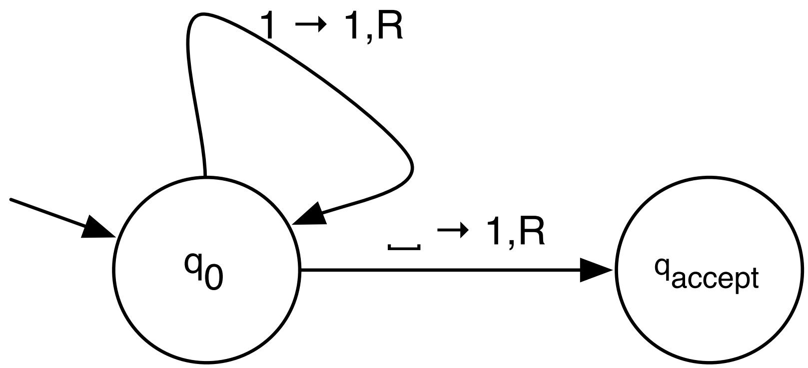
- Technical "Trick"
  - When we reach a halting state h, we delete the configuration (one symbol at a time) until it disappears
  - For each halting state h, and each symbol x, we need



  - Finally,



Example: w=11



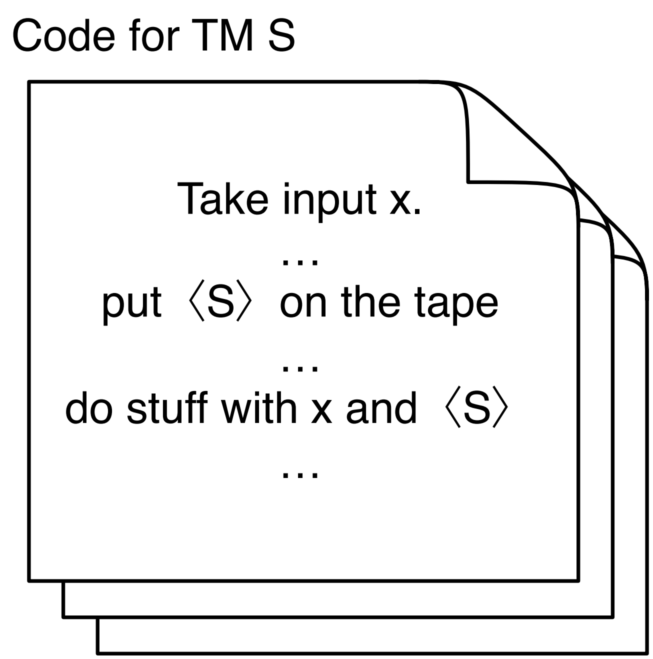
#	1	$\sqcup$	#	#
$\#q_0 11\#$	1	$\sqcup$	#	$\sqcup\#$

$q_0 1$	$q_0 \sqcup$	$1 q_a$	$q_a 1$	$q_a \#\#$
$1 q_0$	$1 q_a$	q_a	q_a	#

The Recursion Theorem

Recursion Theorem (Kleene 1938)

- **Informally**: A program can have access to its own description (code).



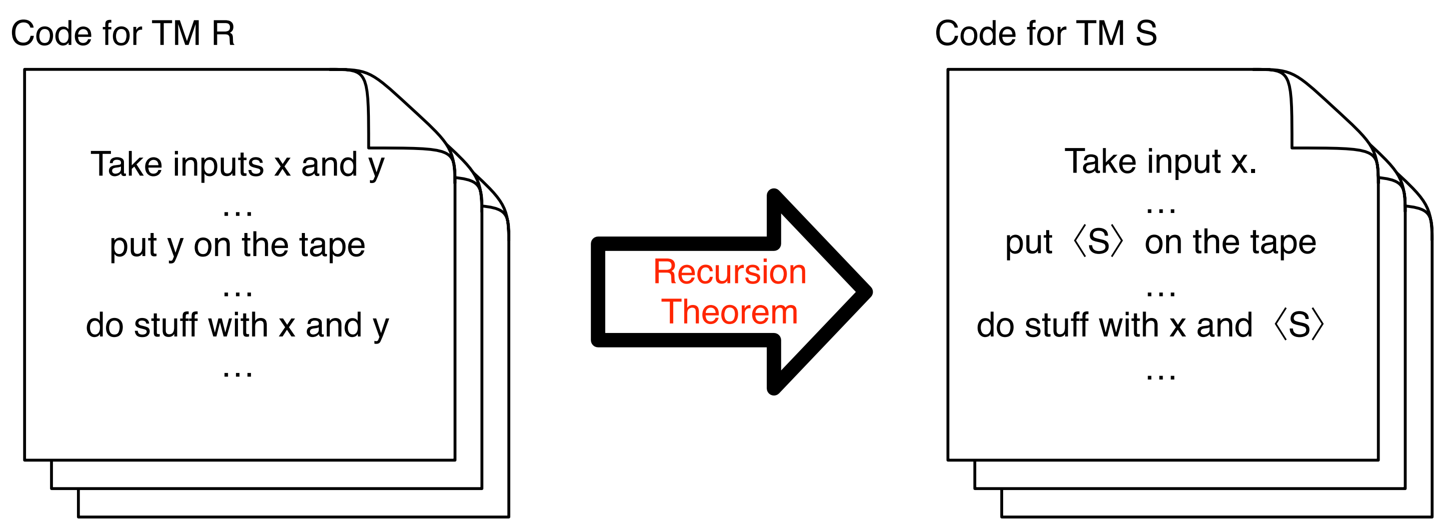
Recursion Theorem (Kleene 1938)

Formally:

If R is a Turing machine computing a binary function R(x, y), then there is a Turing machine S computing a unary function such that:

$$S(x) = R(x, \langle S \rangle)$$

where $\langle S \rangle$ is the description of S itself.



Application: Undecidability of A_{TM} (again)

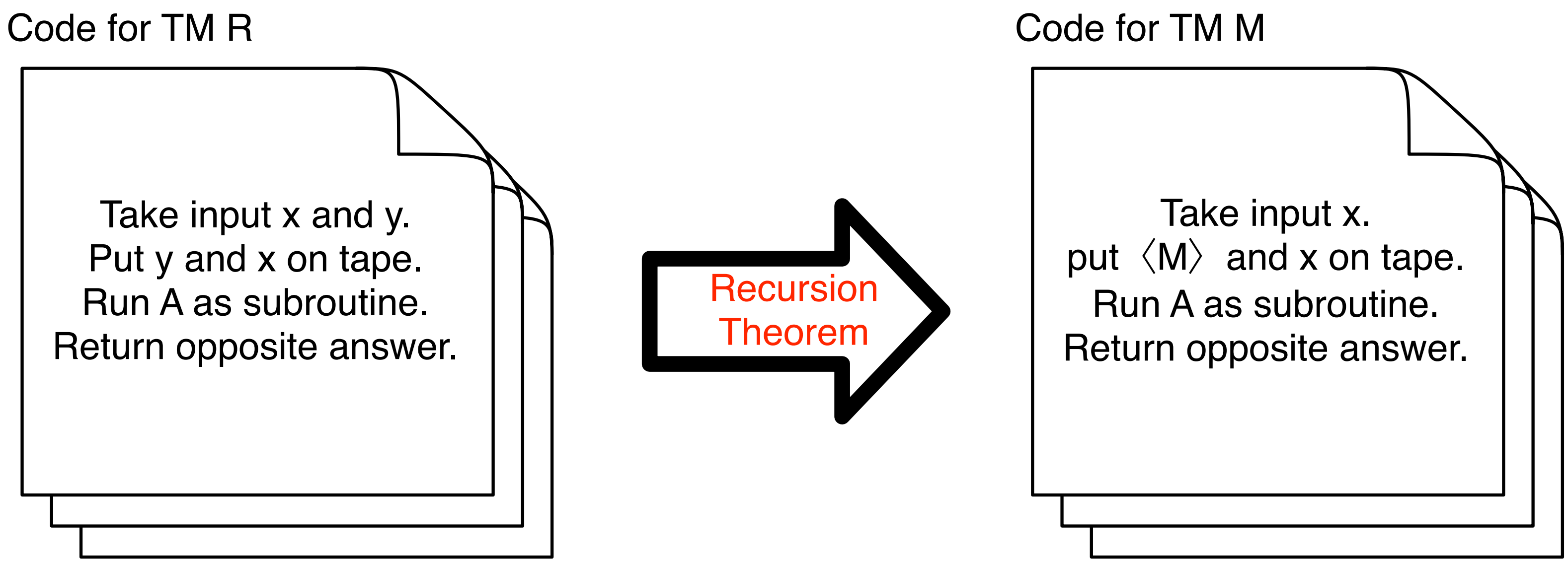
- Suppose A_{TM} were decidable using TM A.

- Define M(x) as follows:

  1. Take input x;
  2. Use A to decide whether M accepts x, i.e., whether
     $\langle M,x \rangle \in A_{TM}$
  3. Return the opposite answer.

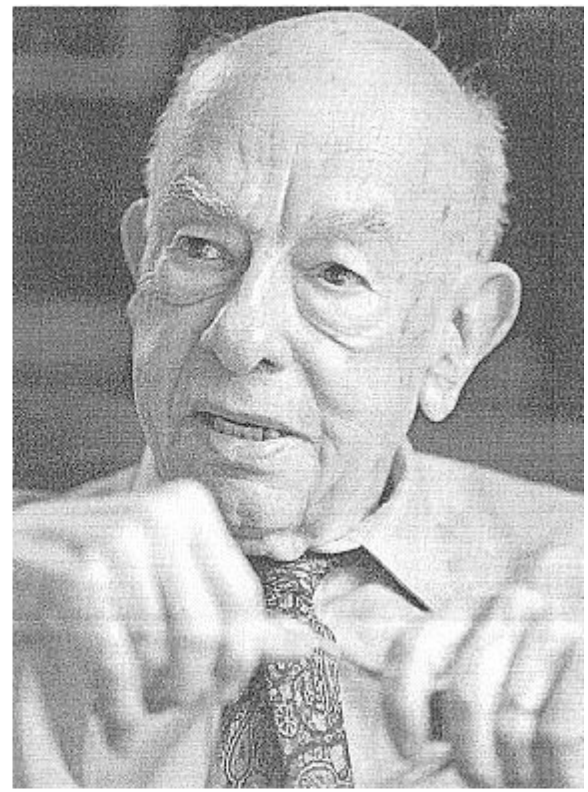
- M can't exist, so A must not exist. QED

Recursion Theorem to the rescue



Self-Printing Machines

- Even if a machine is not given a handle to its own code on its tape at the outset, there are ways for it to construct it.
- Such programs are now called "Quines"



Willard Van Orman Quine 1908-2000

A Java Quine
http://www.knet.ro/lsantha/

```
class Q{public static void main(String[]v){char
c=34;System.out.print(s+c+s+c+';'+'}');}static String
s="class Q{public static void main(String[]v){char
c=34;System.out.print(s+c+s+c+';'+'}');}static String
s=";}
```

- javac Q.java
- java Q

class Q{public static void main(String[]v){char
c=34;System.out.print(s+c+s+c+';'+'}');}static String
s="class Q{public static void main(String[]v){char
c=34;System.out.print(s+c+s+c+';'+'}');}static String
s=";}

C and C++ Quines (authors unknown)

```
char f[] =
"char f[] =%c%c%s%c;%cmain() {printf(f,10,34,f,34,10,10);}%c";
main() {printf(f,10,34,f,34,10,10);}
```

```
#include <iostream>
#define a(b) std::cout<<"#include <iostream>\n#define a(b) "<<#b<<"\nmain(){a("<<#b<<");}"
main(){a(std::cout<<"#include <iostream>\n#define a(b) "<<#b<<"\nmain(){a("<<#b<<");}");}
```

rex Quine
(by a Pomona College Student)

```
a="\\";aa="a";
b="\"";bb="b";
c="=";cc="c";
d="print(

                  aa,c,     b,   a,a,b,    f,
                 aa,aa,    c,b,aa,b,f,g,bb,c,b,
                a,b,b,f   ,bb,bb,c,b,bb,b,f,g,
               cc,c,b,c  ,b,f,  cc,cc,c,b,cc,
              b,   f,g  ,dd           ,c,b,
             d,    b,f  ,              g,g,
            dd,    dd,                  c,b,
            dd,    b,f                ,g,ee
           ,c,b     ,e,              b,f,
           ee,      ee,              c,b,
          ee,b,f,   g,ff,c,           b,f,
          b,f,ff,ff,c,b,ff,b,f,      g,gg,
                c,b,a        ,nn,b,
                f,gg        ,gg,c,b,
                gg,b,f,      g,nn,nn,c
               ,b,nn,b,     f,g,g,d,g);";
```

continued next col.

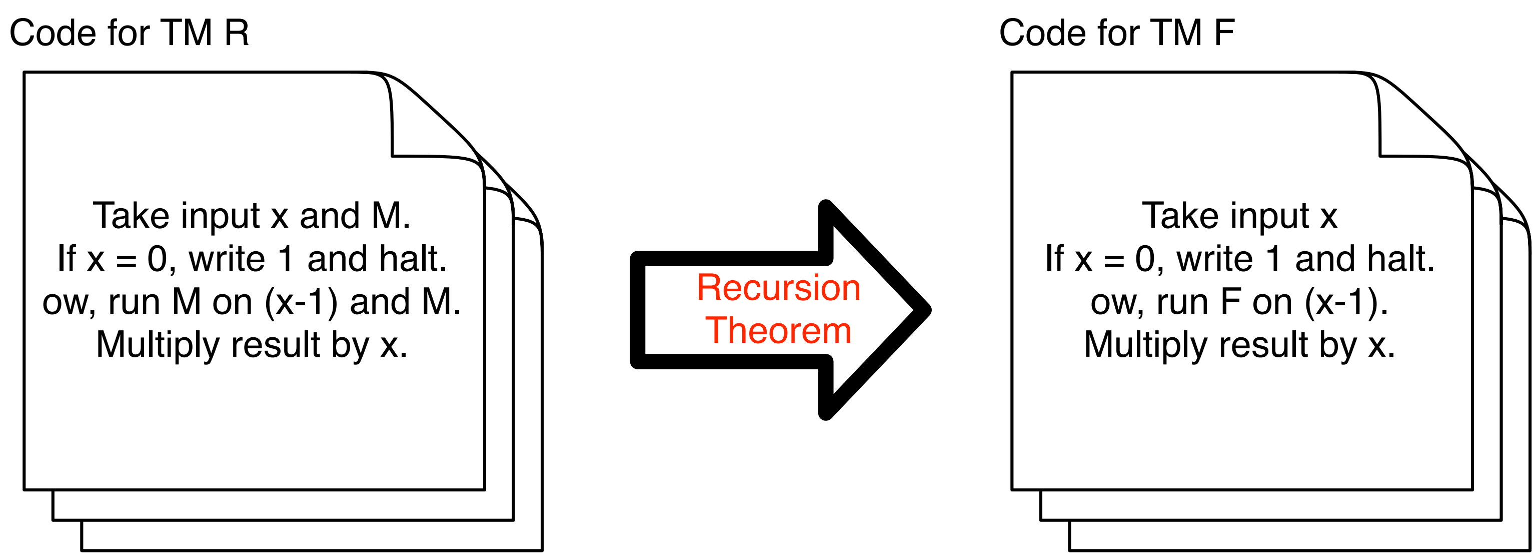
```
dd="d";
e=")";ee="e";
f=";";ff="f";
g="\n";gg="g";
nn="n";

print(

                  aa,c,     b,   a,a,b,    f,
                 aa,aa,    c,b,aa,b,f,g,bb,c,b,
                a,b,b,f   ,bb,bb,c,b,bb,b,f,g,
               cc,c,b,c  ,b,f,  cc,cc,c,b,cc,
              b,   f,g  ,dd           ,c,b,
             d,    b,f  ,              g,g,
            dd,    dd,                  c,b,
            dd,    b,f                ,g,ee
           ,c,b     ,e,              b,f,
           ee,      ee,              c,b,
          ee,b,f,   g,ff,c,           b,f,
          b,f,ff,ff,c,b,ff,b,f,      g,gg,
                c,b,a        ,nn,b,
                f,gg        ,gg,c,b,
                gg,b,f,      g,nn,nn,c
               ,b,nn,b,     f,g,g,d,g);
```

Applications of Quines

- Entertainment
- Computer viruses
- Artificial life?

Recursion from the Recursion Theorem

- ✓ If you have ⟨M⟩ on the tape, you can run it.
- ✓ A machine M can put ⟨M⟩ on the tape.
- ✓ Therefore, a TM can compute “recursively” in the modern sense.



Theorem (Sipser)

- The language MIN = { $\langle M \rangle$ | M is minimal } is not even <u>recognizable</u>.

Minimal

=

no machine with the same behavior has smaller description.

Proof

- Suppose MIN **were** recognizable, hence enumerated by some machine E.
- Consider the TM C:
  - Take input x.
  - Start running enumerator E.
  - Stop when we produce a $\langle D \rangle$ that is **strictly longer** than $\langle C \rangle$ (This must happen. Why?)
  - Simulate D running on x.
- C cannot exist (why?) , so E cannot exist.
- Thus MIN cannot be recognizable.