Tableau and Sequent Calculus for Predicate Logic

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Aside: Additional Tableau Rules: $\varphi \leftrightarrow \psi$

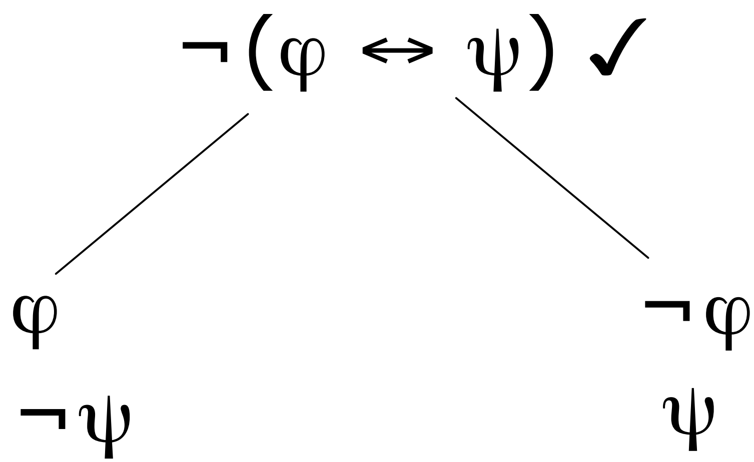
- This formula is retired and the tree **branches** with the formulas and their negations stacked:

$$\varphi \leftrightarrow \psi \checkmark$$

φ	$\neg\varphi$
ψ	$\neg\psi$

Additional Tableau Rules: $\neg(\varphi \leftrightarrow \psi)$

- This formula is retired and the tree **branches**, with φ negated:



Propositional Tableau Rule Summary

stack	$\varphi \wedge \psi$ φ ψ	$\neg(\varphi \vee \psi)$ $\neg\varphi$ $\neg\psi$	$\neg(\varphi \rightarrow \psi)$ φ $\neg\psi$		$\neg\neg\varphi$ φ
split	$\varphi \vee \psi$ φ ψ	$\neg(\varphi \wedge \psi)$ $\neg\varphi$ $\neg\psi$	$\varphi \rightarrow \psi$ $\neg\varphi$ ψ	$(\varphi \leftrightarrow \psi)$ φ $\neg\varphi$ ψ $\neg\psi$ $\neg(\varphi \leftrightarrow \psi)$ φ $\neg\varphi$ $\neg\psi$ ψ	

Quantifier Rules for Tableaux

¬∃ rule

$\neg \exists \nu\ \varphi$ ✓

$\forall \nu\ \neg \varphi$

¬∀ rule

$\neg\forall\nu\ \varphi$ ✓

$\exists\nu\ \neg\varphi$

∃ rule

$$\exists\nu\ \varphi\ \checkmark$$

$$\varphi[c/\nu]$$

where c is a **new** constant not appearing in the tree.

This rule can be used only **once** per path per ∃ formula.

∀ rule

$\forall\nu\ \varphi$ **Does not get a check!!**

$\varphi[\tau/\nu]$

where τ is any term free to replace ν in φ.

This rule can used arbitrarily-many times for a ∀ formula.

Example: Prove $\exists x \, \forall y \, p(x, y) \mid\!- \forall y \exists x \, p(x, y)$

1.	$\exists x \, \forall y \, p(x, y)$	LHS
2.	$\neg \forall y \exists x \, p(x, y)$	RHS negated

Example: Prove ∃x ∀y p(x, y) |– ∀y∃x p(x, y)

1.	∃x ∀y p(x, y)		LHS
2.	¬∀y∃x p(x, y)	✓	RHS negated
3.	∃y¬∃x p(x, y)		2, ¬∀ rule

Example: Prove ∃x ∀y p(x, y) |– ∀y∃x p(x, y)

1.	∃x ∀y p(x, y)	✓	LHS
2.	¬∀y∃x p(x, y)	✓	RHS negated
3.	∃y¬∃x p(x, y)		2, ¬∀ rule
4.	∀y p(a, y)		1, ∃ rule

Example: Prove ∃x ∀y p(x, y) |– ∀y∃x p(x, y)

1.	∃x ∀y p(x, y)	✓	LHS
2.	¬∀y∃x p(x, y)	✓	RHS negated
3.	∃y¬∃x p(x, y)	✓	2, ¬∀ rule
4.	∀y p(a, y)		1, ∃ rule
5.	¬∃x p(x, b)		3, ∃ rule
6.	p(a, b)		4, ∀ rule

Example: Prove ∃x ∀y p(x, y) |– ∀y∃x p(x, y)

1.	∃x ∀y p(x, y)	✓	LHS
2.	¬∀y∃x p(x, y)	✓	RHS negated
3.	∃y¬∃x p(x, y)	✓	2, ¬∀ rule
4.	∀y p(a, y)		1, ∃ rule
5.	¬∃x p(x, b)	✓	3, ∃ rule
6.	p(a, b)		4, ∀ rule
7.	∀x ¬p(x, b)		5, ¬∃ rule

Example: Prove ∃x ∀y p(x, y) |– ∀y∃x p(x, y)

1.	∃x ∀y p(x, y)	✓	LHS
2.	¬∀y∃x p(x, y)	✓	RHS negated
3.	∃y¬∃x p(x, y)	✓	2, ¬∀ rule
4.	∀y p(a, y)		1, ∃ rule
5.	¬∃x p(x, b)	✓	3, ∃ rule
6.	p(a, b)		4, ∀ rule
7.	∀x ¬p(x, b)		5, ¬∃ rule
8.	¬p(a, b)		7, ∀ rule
	X closes (6, 8)		

Example: Prove $\exists x\ \forall y\ p(x, y) \mid\!- \forall y \exists x\ p(x, y)$

1.	$\exists x\ \forall y\ p(x, y)$	✓	LHS
2.	$\neg\forall y\exists x\ p(x, y)$	✓	RHS negated
3.	$\exists y\neg\exists x\ p(x, y)$	✓	2, $\neg\forall$ rule
4.	$\forall y\ p(a, y)$		1, $\exists$ rule
5.	$\neg\exists x\ p(x, b)$	✓	3, $\exists$ rule
6.	$p(a, b)$		4, $\forall$ rule
7.	$\forall x\ \neg p(x, b)$		5, $\neg\exists$ rule
8.	$\neg p(a, b)$		7, $\forall$ rule
	X closes (6, 8)		

Corresponding natural deduction proof

1:	$\exists x.\forall y.P(x,y)$	premise
2:	actual i	assumption
3:	actual i1, $\forall y.P(i1,y)$	assumptions
4:	$P(i1,i)$	$\forall$ elim 3.2,2
5:	$\exists x.P(x,i)$	$\exists$ intro 4,3.1
6:	$\exists x.P(x,i)$	$\exists$ elim 1,3–5
7:	$\forall y.\exists x.P(x,y)$	$\forall$ intro 2–6

Example
(in which ∀ rule is used twice from the same line)

$\forall x\, (\exists y\, P(x,y) \rightarrow \forall z\, P(z, x)),\ P(a,a)\ \vdash P(a, b)$

This time we number for better clarity.

1. $\forall x\, (\exists y\, P(x,y) \rightarrow \forall z\, P(z, x))$ premise
2. $P(a,a)$ premise
3. $\neg P(a, b)$ negated conclusion
4. $(\exists y\, P(a,y) \rightarrow \forall z\, P(z, a))$ √ 1 with a for x

5.1 $\neg\exists y\, P(a,y)$ √ 5.2 $\forall z\, P(z, a)$

6.1 $\forall y \neg P(a, y)$ 6.2 $P(b, a)$ 5.2 with b for z

7.1 $\neg P(a, a)$ 7.2 $\exists y\, P(b,y) \rightarrow \forall z\, P(z, b)$ 1 with b for x

X closes (2, 7.1)

8.2.1 $\neg\exists y\, P(b,y)$ 6.2.2 $\forall z\, P(z, b)$

9.2.1 $\forall y\, \neg P(b,y)$ 7.2.2 $P(a, b)$

10.2.1 $\neg P(b,a)$ 8.2.2 **X closes** (3, 7.2.2)

X closes (6.2, 10.2.1)

Choice of Terms for $\forall$ rule

- Prefer terms constructed of constants introduced earlier [as they are more likely to produce closure].

- Introduce additional new constants as needed.

- Adding new constants invokes the **non-empty domain assumption** implicitly.

Example (already negated)

$$\neg(\forall x\ p(x) \rightarrow \exists x\ p(x))$$

Example

$$\neg(\forall x\ p(x) \rightarrow \exists x\ p(x)) \checkmark$$

$$\forall x\ p(x)$$

$$\neg\exists x\ p(x)$$

Example

$$\neg(\forall x\ p(x) \rightarrow \exists x\ p(x)) ✓$$

$$\forall x\ p(x)$$

$$\neg\exists x\ p(x)\ ✓$$

$$\forall x\ \neg p(x)$$

To proceed further, we introduce a new constant ‘a’, then use the ∀x rule (twice).

Example

$\neg(\forall x\ p(x) \rightarrow \exists x\ p(x))$ ✓

$\forall x\ p(x)$

$\neg\exists x\ p(x)$ ✓

$\forall x\ \neg p(x)$

$p(a)$

$\neg p(a)$

X closed

∀

a is new

∀

The root formula is not satisfiable.

Thus $\forall x\ p(x) \rightarrow \exists x\ p(x)$ is valid

Closure depended on appropriate choice of term to substitute for x in $\forall x\ \neg p(x)$.

Termination

- Unlike the propositional case, the predicate version of tableaux **does not necessarily terminate**. This is because the ∀ rule can be used arbitrarily-many times.

- It can be shown, however, that **if** the root formula is **un**satisfiable, then **there exists** a closed tree for it.

- (If the root formula **is** satisfiable, the construction *might* not terminate.)

Example of Non-Termination

$$\forall x\ \exists y\ P(x, y) \vdash P(a, a)$$

1.	$\forall x\ \exists y\ P(x, y)$	premise
2.	$\neg\ P(a, a)$	conclusion negated
3.	$\exists y\ P(a, y)$ √	1, a for x
4.	$P(a, b)$	3, b for y
5.	$\exists y\ P(b, y)$ √	1, b for x
6.	$P(b, c)$	5, c for y
7.	$\exists y\ P(c, y)$ √	1, c for x
8.	$P(c, d)$	7, d for y
9.	$\exists y\ P(d, y)$ √	1, d for x
	. . .	. . .

Using the Tableau Method to Find a Model in the Predicate Calculus

- $\forall x\ (A(x) \vee B(x)) \rightarrow (\forall x\ A(x) \vee \forall x\ B(x))$
- This formula is not valid, so its negation should be satisfiable.

$\neg(\forall x\ (A(x) \vee B(x)) \rightarrow (\forall x\ A(x) \vee \forall x\ B(x)))$ ✓

$\forall x\ (A(x) \vee B(x))$

$\neg(\forall x\ A(x) \vee \forall x\ B(x))$ ✓

$\neg\forall x\ A(x)$ ✓

$\neg\forall x\ B(x)$ ✓

$\exists x\ \neg A(x)$

$\exists x\ \neg B(x)$

Using the Tableau Method to Find a Model in the Predicate Calculus

$\forall x\,(A(x) \vee B(x))$

$\exists x\, \neg A(x)$ ✓

$\exists x\, \neg B(x)$ ✓

$\neg A(a)$

$\neg B(b)$

$A(a) \vee B(a)$ ✓

$A(b) \vee B(b)$ ✓

$A(a)$
X closes

$B(a)$

$A(b)$
won't close

$B(b)$
X closes

Conclusion: **There is a model for the negation** with domain {a, b}, in which "¬A(a), A(b), B(a), and ¬B(b)" (translated into the appropriate interpretation notation).

Check for Counterexample

- A model that shows the negation is satisfiable is a counterexample for the validity of the original.

- $\forall x\ (A(x) \vee B(x)) \rightarrow (\forall x\ A(x) \vee \forall x\ B(x))$

- $\neg A(a)$, $A(b)$, $B(a)$, and $\neg B(b)$
- Domain = {a, b}

- $I[\forall x\ (A(x) \vee B(x))] = T$, but
  $I[\forall x\ A(x) \vee \forall x\ B(x)] = F$.

Handling Equality in Tableaux

- If an open path has a node $t_1 = t_2$, then for any unchecked node φ containing t_1, add on the path a formula in which t_1 is replaced with t_2 (and vice-versa), so long as appropriate rules for substitution are observed.

Example: Equality in Tableau

- P(a)∨P(b), ¬P(a) |— ¬(a=b)
- Proof:

1. P(a)∨P(b)	premise
2. ¬P(a)	premise
3. ¬¬(a=b) ✓	negated conclusion
4. a=b	3, ¬¬
5. ¬P(b)	2, 4, = rule

P(a)   P(b)   1

X closes   X closes

Example: Equality in Tableau

- $a = b \vdash P(a, b) \rightarrow P(b, a)$
- Proof:

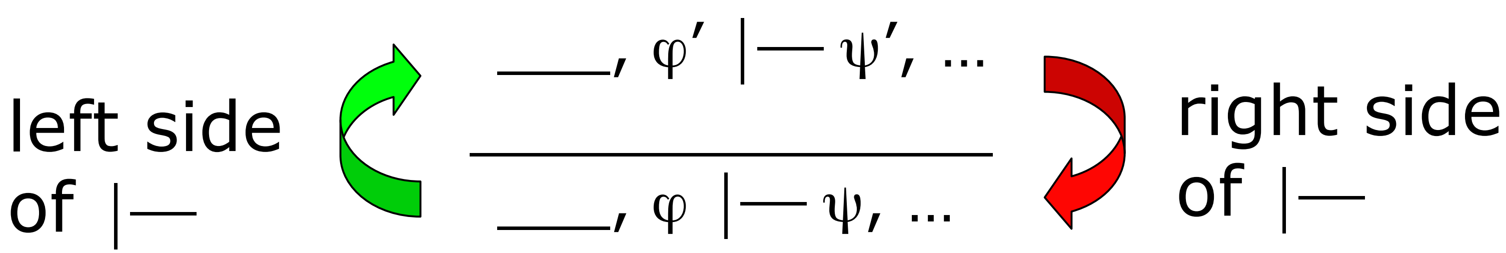
1. $a = b$	premise
2. $\neg(P(a, b) \rightarrow P(b, a))$ ✓	negated conclusion
3. $P(a, b)$	2
4. $\neg P(b, a)$	2
5. $P(a, a)$	3, 1, = rule
6. $\neg P(a, a)$	4, 1, = rule
X closes(5, 6)	

Sequent Calculus for Predicates

- As with the tableau method, the propositional rules for Sequent Calculus will be augmented with four new rules for quantifiers.
- As before, the Sequent Calculus rules have a correspondence with the tableau proof rules.
- Whereas the tableau shows negation explicitly, in Sequent Calculus it may be implicit, depending on which side of the turnstile a formula appears.
- **Negated formulas in tableaux** generally correspond to **formulas on the right** of the turnstile in Sequent Calculus.

Mindset for Remembering Sequent Calculus

- Think of the sequent(s) above the line as being **sufficient** to prove the one below.
- Think of the "information flow" as shown in the diagram.



∀L rule

$$\frac{\_\_\_\_,\ \forall x\ \varphi,\ \varphi[t/x]\ \vdash\ \ldots}{\_\_\_\_,\ \forall x\ \varphi\ \vdash\ \ldots}\ \forall L$$

where t is any term.

This rule parallels the **∀-Elimination** rule of natural deduction.

It says that … can be proved from ____, ∀x φ provided that it can be proved from ____, ∀x φ, $\varphi[t/x]$.

∃L rule

$$\frac{\_\_\_\_,\ \varphi[x_0/x]\ \mid\!\!-\ \ldots}{\_\_\_\_,\ \exists x\ \varphi\ \mid\!\!-\ \ldots}\ \exists L$$

where x_0 is a fresh variable not occurring in ____ or … .

This rule parallels the **∃-Elimination** rule of natural deduction.

It says that … can be proved from ____, $\exists x\varphi$ provided that it can be proved from ____, $\varphi[x_0/x]$.

∀R rule

$$\frac{\_\_\_\_ \;|\!\!-\!\!- \;\varphi[x_0/x], \dots}{\_\_\_\_ \;|\!\!-\!\!- \;\forall x\, \varphi, \dots} \;|\!\!-\!\!- \;\forall R$$

where x_0 is a fresh variable not occurring in ____ or … .

This rule parallels the **∀-Introduction** rule of natural deduction.

It states that to prove $\forall x \varphi$ it suffices to prove $\varphi[x_0/x]$ where x_0 is an an arbitrary fresh variable.

∃R rule

$$\frac{\_\_\_\_ \vdash \varphi[t/x], \exists x\ \varphi, \ldots}{\_\_\_\_ \vdash \exists x\ \varphi, \ldots} \vdash \exists R$$

where t is an term free for x in φ.

This rule parallels the **∃-Introduction rule** of natural deduction.

It states that to prove $\exists x \varphi$ it suffices to prove $\varphi[t/x]$ where t is any term.

Sequent Calculus Quantifier Rule Summary

	Left	Right
$\exists$	$\dfrac{\_\_\_\_,\ \varphi[x_0/x]\ \vdash\ \ldots}{\_\_\_\_,\ \exists x\ \varphi\ \vdash\ \ldots}\ \exists L$	$\dfrac{\_\_\_\_\ \vdash\ \varphi[t/x],\ \exists x\ \varphi,\ \ldots}{\_\_\_\_\ \vdash\ \exists x\ \varphi,\ \ldots}\ \vdash \exists R$
$\forall$	$\dfrac{\_\_\_\_,\ \forall x\ \varphi,\ \varphi[t/x]\ \vdash\ \ldots}{\_\_\_\_,\ \forall x\ \varphi\ \vdash\ \ldots}\ \forall L$	$\dfrac{\_\_\_\_\ \vdash\ \varphi[x_0/x],\ \ldots}{\_\_\_\_\ \vdash\ \forall x\ \varphi,\ \ldots}\ \vdash \forall R$

Sequent Calculus Example

$$\exists x\ \forall y\ p(x, y) \mid\!- \forall y \exists x\ p(x, y)$$

Sequent Calculus Example

$$\frac{\_\_\_\_,\ \varphi[x_0/x]\ \vdash\ \ldots}{\_\_\_\_,\ \exists x\ \varphi\ \vdash\ \ldots}\ \exists L$$

$$\exists x\ \forall y\ p(x, y) \vdash \forall y \exists x\ p(x, y)$$

Sequent Calculus Example

$$\frac{\forall y\ p(a, y) \vdash \forall y \exists x\ p(x, y)}{\exists x\ \forall y\ p(x, y) \vdash \forall y \exists x\ p(x, y)}\ \exists L$$

Sequent Calculus Example

$$\frac{\_\_\_\_ \vdash \varphi[x_0/x], \ldots}{\_\_\_\_ \vdash \forall x\ \varphi, \ldots} \vdash \forall R$$

$$\frac{\forall y\ p(a, y) \vdash \forall y \exists x\ p(x, y)}{\exists x\ \forall y\ p(x, y) \vdash \forall y \exists x\ p(x, y)} \quad \exists L$$

Sequent Calculus Example

$$\dfrac{\dfrac{\forall y\ p(a, y) \vdash \exists x\ p(x, b)}{\forall y\ p(a, y) \vdash \forall y \exists x\ p(x, y)}\ \forall R}{\exists x\ \forall y\ p(x, y) \vdash \forall y \exists x\ p(x, y)}\ \exists L$$

Sequent Calculus Example

$$\frac{\_\_\_\_,\ \forall x\ \varphi,\ \varphi[t/x]\ \vdash \ldots}{\_\_\_\_,\ \forall x\ \varphi\ \vdash \ldots}\ \forall L$$

$$\cfrac{\cfrac{\forall y\ p(a, y) \vdash \exists x\ p(x, b)}{\forall y\ p(a, y) \vdash \forall y \exists x\ p(x, y)}\ \forall R}{\exists x\ \forall y\ p(x, y) \vdash \forall y \exists x\ p(x, y)}\ \exists L$$

Sequent Calculus Example

$$\forall y\ p(a, y),\ p(a, b) \vdash \exists x\ p(x, b)$$

$$\forall y\ p(a, y) \vdash \exists x\ p(x, b) \qquad \forall L$$

$$\forall y\ p(a, y) \vdash \forall y \exists x\ p(x, y) \qquad \forall R$$

$$\exists x\ \forall y\ p(x, y) \vdash \forall y \exists x\ p(x, y) \qquad \exists L$$

Sequent Calculus Example

$$\frac{\_\_\_ \vdash \varphi[t/x], \ldots}{\_\_\_ \vdash \exists x\, \varphi, \ldots} \vdash \exists R$$

$$\frac{\dfrac{\dfrac{\forall y\, p(a, y),\ p(a, b) \vdash \exists x\, p(x, b)}{\forall y\, p(a, y) \vdash \exists x\, p(x, b)}\ \forall L}{\forall y\, p(a, y) \vdash \forall y \exists x\, p(x, y)}\ \forall R}{\exists x\, \forall y\, p(x, y) \vdash \forall y \exists x\, p(x, y)}\ \exists L$$

Sequent Calculus Example

$\forall y\ p(a, y), p(a, b) \vdash p(a, b)$

$\forall y\ p(a, y), p(a, b) \vdash \exists x\ p(x, b)$ ∃R

$\forall y\ p(a, y) \vdash \exists x\ p(x, b)$ ∀L

$\forall y\ p(a, y) \vdash \forall y \exists x\ p(x, y)$ ∀R

$\exists x\ \forall y\ p(x, y) \vdash \forall y \exists x\ p(x, y)$ ∃L

Sequent Calculus Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall y\ p(a, y),\ p(a, b) \vdash p(a, b)}\ \text{Ax}}{\forall y\ p(a, y),\ p(a, b) \vdash \exists x\ p(x, b)}\ \exists R}{\forall y\ p(a, y) \vdash \exists x\ p(x, b)}\ \forall L}{\forall y\ p(a, y) \vdash \forall y \exists x\ p(x, y)}\ \forall R}{\exists x\ \forall y\ p(x, y) \vdash \forall y \exists x\ p(x, y)}\ \exists L$$

Sequent Calculus vs. Tableau

Sequent Calculus

$\forall y\, p(a, y), p(a, b) \vdash p(a, b)$	Ax
$\forall y\, p(a, y), p(a, b) \vdash \exists x\, p(x, b)$	$\exists$R
$\forall y\, p(a, y) \vdash \exists x\, p(x, b)$	$\forall$L
$\forall y\, p(a, y) \vdash \forall y \exists x\, p(x, y)$	$\forall$R
$\exists x\, \forall y\, p(x, y) \vdash \forall y \exists x\, p(x, y)$	$\exists$L

Block Tableau Inverted

$\{\forall y\, p(a, y), p(a, b), \neg p(a, b)\}$
$\{\forall y\, p(a, y), p(a, b), \forall x\, \neg p(x, b)\}$
$\{\forall y\, p(a, y), p(a, b), \neg \exists x\, p(x, b)\}$
$\{\forall y\, p(a, y), \neg \exists x\, p(x, b)\}$
$\{\forall y\, p(a, y), \exists y \neg \exists x\, p(x, y)\}$
$\{\exists x\, \forall y\, p(x, y), \exists y \neg \exists x\, p(x, y)\}$
$\{\exists x\, \forall y\, p(x, y), \neg \forall y \exists x\, p(x, y)\}$

Block Tableau

$\{\exists x\, \forall y\, p(x, y), \neg \forall y \exists x\, p(x, y)\}$
$\{\exists x\, \forall y\, p(x, y), \exists y \neg \exists x\, p(x, y)\}$
$\{\forall y\, p(a, y), \exists y \neg \exists x\, p(x, y)\}$
$\{\forall y\, p(a, y), \neg \exists x\, p(x, b)\}$
$\{\forall y\, p(a, y), p(a, b), \neg \exists x\, p(x, b)\}$
$\{\forall y\, p(a, y), p(a, b), \forall x\, \neg p(x, b)\}$
$\{\forall y\, p(a, y), p(a, b), \neg p(a, b)\}$

Tableau

1.	$\exists x\, \forall y\, p(x, y)$	✓	LHS
2.	$\neg \forall y \exists x\, p(x, y)$	✓	RHS negated
3.	$\exists y \neg \exists x\, p(x, y)$	✓	2, $\neg\forall$ rule
4.	$\forall y\, p(a, y)$		1, $\exists$ rule
5.	$\neg \exists x\, p(x, b)$	✓	3, $\exists$ rule
6.	$p(a, b)$		4, $\forall$ rule
7.	$\forall x\, \neg p(x, b)$		5, $\neg\exists$ rule
8.	$\neg p(a, b)$		7, $\forall$ rule
	X closes (6, 8)		

Rule Correspondence

Tableau Rule (going downward)	Sequent Calculus Rule (going backward/upward)
¬∃ replace with ∀¬	∃R specializes term on right
¬∀ replace with ∃¬	∀R specializes term on right to fresh var
∃ introduces constant	∃L specializes term on left to fresh var
∀ uses term	∀L specializes term on left
closing path	axiom

http://www.umsu.de/logik/trees/

Proof Example: $\vdash \exists y\,(A(y) \rightarrow \forall z\, A(z))$

1. $\neg\exists y(Ay \rightarrow \forall z Az)$
2. $\neg(Aa \rightarrow \forall z Az)$ (1)
3. Aa (2)
4. $\neg\forall z Az$ (2)
5. $\neg Ab$ (4)
6. $\neg(Ab \rightarrow \forall z Az)$ (1)
7. Ab (6)
8. $\neg\forall z Az$ (6)

x

Two uses of (1)