Tableau and Sequent Calculus for Predicate Logic

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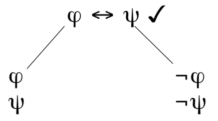
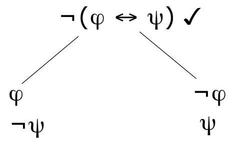
Aside: Additional Tableau Rules: φ ↔ ψ

- This formula is retired and the tree **branches** with the formulas and their negations stacked:



Additional Tableau Rules: ¬(φ ↔ ψ)

- This formula is retired and the tree **branches**, with φ negated:



Propositional Tableau Rule Summary

stack	φ ∧ ψ φ ψ	¬(φ ∨ ψ) ¬φ ¬ψ	¬(φ→ψ) φ ¬ψ		¬¬φ φ
split	φ ∨ ψ φ ψ	¬(φ ∧ ψ) ¬φ ¬ψ	φ → ψ ¬φ ψ	(φ ↔ ψ) φ ¬φ ψ ¬ψ ¬(φ ↔ ψ) φ ¬φ ¬ψ ψ	

Quantifier Rules for Tableaux

¬∃ rule

¬∃v φ ✓
∀v ¬φ

$\neg\forall$ rule

$\neg\forall\nu\ \varphi$ ✓

$\exists\nu\ \neg\varphi$

$\exists$ rule

$\exists\nu\ \varphi$ ✓

$\varphi[c/\nu]$

where c is a **new** constant not appearing in the tree.

This rule can be used only **once** per path per $\exists$ formula.

$\forall$ rule

$\forall\nu\ \varphi$ **Does not get a check!!**

$\varphi[\tau/\nu]$

where τ is any term free to replace ν in φ.

This rule can used arbitrarily-many times for a $\forall$ formula.

Example: Prove $\exists x\ \forall y\ p(x, y) \vdash \forall y \exists x\ p(x, y)$

1.	$\exists x\ \forall y\ p(x, y)$	LHS
2.	$\neg\forall y\exists x\ p(x, y)$	RHS negated

Example: Prove $\exists x\ \forall y\ p(x, y) \vdash \forall y \exists x\ p(x, y)$

1.	$\exists x\ \forall y\ p(x, y)$		LHS
2.	$\neg\forall y\exists x\ p(x, y)$	✓	RHS negated
3.	$\exists y\neg\exists x\ p(x, y)$		2, $\neg\forall$ rule

Example: Prove $\exists x\ \forall y\ p(x, y) \vdash \forall y \exists x\ p(x, y)$

1.	$\exists x\ \forall y\ p(x, y)$	✓	LHS
2.	$\neg\forall y\exists x\ p(x, y)$	✓	RHS negated
3.	$\exists y\neg\exists x\ p(x, y)$		2, $\neg\forall$ rule
4.	$\forall y\ p(a, y)$		1, $\exists$ rule

Example: Prove ∃x ∀y p(x, y) |– ∀y∃x p(x, y)

1.	∃x ∀y p(x, y)	✓	LHS
2.	¬∀y∃x p(x, y)	✓	RHS negated
3.	∃y¬∃x p(x, y)	✓	2, ¬∀ rule
4.	∀y p(a, y)		1, ∃ rule
5.	¬∃x p(x, b)		3, ∃ rule
6.	p(a, b)		4, ∀ rule

Example: Prove ∃x ∀y p(x, y) |– ∀y∃x p(x, y)

1.	∃x ∀y p(x, y)	✓	LHS
2.	¬∀y∃x p(x, y)	✓	RHS negated
3.	∃y¬∃x p(x, y)	✓	2, ¬∀ rule
4.	∀y p(a, y)		1, ∃ rule
5.	¬∃x p(x, b)	✓	3, ∃ rule
6.	p(a, b)		4, ∀ rule
7.	∀x ¬p(x, b)		5, ¬∃ rule

Example: Prove ∃x ∀y p(x, y) |– ∀y∃x p(x, y)

1.	∃x ∀y p(x, y)	✓	LHS
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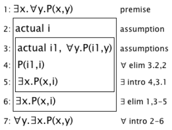
3.	∃y¬∃x p(x, y)	✓	2, ¬∀ rule
4.	∀y p(a, y)		1, ∃ rule
5.	¬∃x p(x, b)	✓	3, ∃ rule
6.	p(a, b)		4, ∀ rule
7.	∀x ¬p(x, b)		5, ¬∃ rule
8.	¬p(a, b)		7, ∀ rule
	X closes (6, 8)		

Example: Prove ∃x ∀y p(x, y) |– ∀y∃x p(x, y)

1.	∃x ∀y p(x, y)	✓	LHS
2.	¬∀y∃x p(x, y)	✓	RHS negated
3.	∃y¬∃x p(x, y)	✓	2, ¬∀ rule
4.	∀y p(a, y)		1, ∃ rule
5.	¬∃x p(x, b)	✓	3, ∃ rule
6.	p(a, b)		4, ∀ rule
7.	∀x ¬p(x, b)		5, ¬∃ rule
8.	¬p(a, b)		7, ∀ rule
	X closes (6, 8)		

Corresponding natural deduction proof



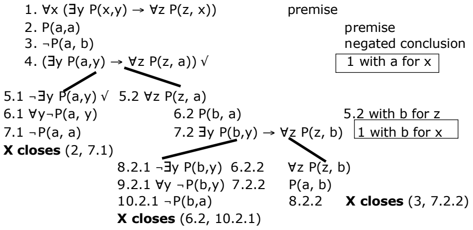
1: ∃x.∀y.P(x,y) premise
2: actual i assumption
3: actual i1, ∀y.P(i1,y) assumptions
4: P(i1,i) ∀ elim 3.2,2
5: ∃x.P(x,i) ∃ intro 4,3.1
6: ∃x.P(x,i) ∃ elim 1,3–5
7: ∀y.∃x.P(x,y) ∀ intro 2–6

Example

(in which ∀ rule is used twice from the same line)

∀x (∃y P(x,y) → ∀z P(z, x)), P(a,a) |— P(a, b)

This time we number for better clarity.



1. ∀x (∃y P(x,y) → ∀z P(z, x)) premise
2. P(a,a) premise
3. ¬P(a, b) negated conclusion
4. (∃y P(a,y) → ∀z P(z, a)) √ 1 with a for x

5.1 ¬∃y P(a,y) √ 5.2 ∀z P(z, a)
6.1 ∀y¬P(a, y) 6.2 P(b, a) 5.2 with b for z
7.1 ¬P(a, a) 7.2 ∃y P(b,y) → ∀z P(z, b) 1 with b for x
X closes (2, 7.1)
8.2.1 ¬∃y P(b,y) 6.2.2 ∀z P(z, b)
9.2.1 ∀y ¬P(b,y) 7.2.2 P(a, b)
10.2.1 ¬P(b,a) 8.2.2 **X closes** (3, 7.2.2)
X closes (6.2, 10.2.1)

Choice of Terms for ∀ rule

- Prefer terms constructed of constants introduced earlier [as they are more likely to produce closure].
- Introduce additional new constants as needed.
- Adding new constants invokes the **non-empty domain assumption** implicitly.

Example (already negated)

¬(∀x p(x) → ∃x p(x))

Example

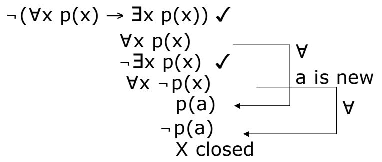
¬(∀x p(x) → ∃x p(x)) ✓
∀x p(x)
¬∃x p(x)

Example

¬(∀x p(x) → ∃x p(x)) ✓
∀x p(x)
¬∃x p(x) ✓
∀x ¬p(x)

To proceed further, we introduce a new constant ‘a’, then use the ∀x rule (twice).

Example



¬(∀x p(x) → ∃x p(x)) ✓
∀x p(x)
¬∃x p(x) ✓
∀x ¬p(x)
p(a)
¬p(a)
X closed

The root formula is not satisfiable.
Thus ∀x p(x) → ∃x p(x) is valid

Closure depended on appropriate choice of term to substitute for x in ∀x ¬p(x).

Termination

- Unlike the propositional case, the predicate version of tableaux **does not necessarily terminate**. This is because the ∀ rule can be used arbitrarily-many times.
- It can be shown, however, that **if** the root formula is **un**satisfiable, then **there exists** a closed tree for it.
- (If the root formula **is** satisfiable, the construction *might* not terminate.)

Example of Non-Termination

∀x ∃y P(x, y) |— P(a, a)

1.	∀x ∃y P(x, y)	premise
2.	¬ P(a, a)	conclusion negated
3.	∃y P(a, y) √	1, a for x
4.	P(a, b)	3, b for y
5.	∃y P(b, y) √	1, b for x
6.	P(b, c)	5, c for y
7.	∃y P(c, y) √	1, c for x
8.	P(c, d)	7, d for y
9.	∃y P(d, y) √	1, d for x
	. . .	. . .

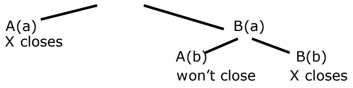
Using the Tableau Method to Find a Model in the Predicate Calculus

- $\forall x\,(A(x) \vee B(x)) \rightarrow (\forall x\,A(x) \vee \forall x\,B(x))$
- This formula is not valid, so its negation should be satisfiable.

$\neg(\forall x\,(A(x) \vee B(x)) \rightarrow (\forall x\,A(x) \vee \forall x\,B(x)))$ ✓
$\forall x\,(A(x) \vee B(x))$
$\neg(\forall x\,A(x) \vee \forall x\,B(x))$ ✓
$\neg\forall x\,A(x)$ ✓
$\neg\forall x\,B(x)$ ✓
$\exists x\,\neg A(x)$
$\exists x\,\neg B(x)$

Using the Tableau Method to Find a Model in the Predicate Calculus

$\forall x\,(A(x) \vee B(x))$
$\exists x\,\neg A(x)$ ✓
$\exists x\,\neg B(x)$ ✓
$\neg A(a)$
$\neg B(b)$
$A(a) \vee B(a)$ ✓
$A(b) \vee B(b)$ ✓


A(a)
X closes
B(a)
A(b)
won't close
B(b)
X closes

Conclusion: **There is a model for the negation** with domain {a, b}, in which "¬A(a), A(b), B(a), and ¬B(b)" (translated into the appropriate interpretation notation).

Check for Counterexample

- A model that shows the negation is satisfiable is a counterexample for the validity of the original.
- $\forall x\,(A(x) \vee B(x)) \rightarrow (\forall x\,A(x) \vee \forall x\,B(x))$
- $\neg A(a)$, $A(b)$, $B(a)$, and $\neg B(b)$
- Domain = {a, b}
- $I[\forall x\,(A(x) \vee B(x))] = T$, but
  $I[\forall x\,A(x) \vee \forall x\,B(x)] = F$.

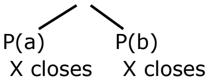
Handling Equality in Tableaux

- If an open path has a node $t_1 = t_2$, then for any unchecked node φ containing t_1, add on the path a formula in which t_1 is replaced with t_2 (and vice-versa), so long as appropriate rules for substitution are observed.

Example: Equality in Tableau

- $P(a) \vee P(b), \neg P(a) \vdash \neg(a=b)$
- Proof:

1. $P(a) \vee P(b)$	premise
2. $\neg P(a)$	premise
3. $\neg\neg(a=b)$ ✓	negated conclusion
4. $a=b$	3, ¬¬
5. $\neg P(b)$	2, 4, = rule


P(a) P(b) 1
X closes X closes

Example: Equality in Tableau

- $a = b \vdash P(a, b) \rightarrow P(b, a)$
- Proof:

1. $a = b$	premise
2. $\neg(P(a, b) \rightarrow P(b, a))$ ✓	negated conclusion
3. $P(a, b)$	2
4. $\neg P(b, a)$	2
5. $P(a, a)$	3, 1, = rule
6. $\neg P(a, a)$	4, 1, = rule

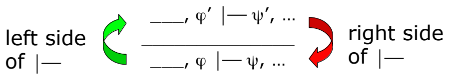
X closes(5, 6)

Sequent Calculus for Predicates

- As with the tableau method, the propositional rules for Sequent Calculus will be augmented with four new rules for quantifiers.
- As before, the Sequent Calculus rules have a correspondence with the tableau proof rules.
- Whereas the tableau shows negation explicitly, in Sequent Calculus it may be implicit, depending on which side of the turnstile a formula appears.
- **Negated formulas in tableaux** generally correspond to **formulas on the right** of the turnstile in Sequent Calculus.

Mindset for Remembering Sequent Calculus

- Think of the sequent(s) above the line as being **sufficient** to prove the one below.
- Think of the “information flow” as shown in the diagram.



left side of |—

$$\frac{\_\_\_\_, \varphi' \vdash \psi', \ldots}{\_\_\_\_, \varphi \vdash \psi, \ldots}$$

right side of |—

∀L rule

$$\frac{\_\_\_\_, \forall x\, \varphi, \varphi[t/x] \vdash \ldots}{\_\_\_\_, \forall x\, \varphi \vdash \ldots}\ \forall L$$

where t is any term.

This rule parallels the **∀-Elimination** rule of natural deduction.

It says that … can be proved from ____, ∀x φ provided that it can be proved from ____, ∀x φ, φ[t/x].

∃L rule

$$\frac{\_\_\_\_, \varphi[x_0/x] \vdash \ldots}{\_\_\_\_, \exists x\, \varphi \vdash \ldots}\ \exists L$$

where x_0 is a fresh variable not occurring in ____ or … .

This rule parallels the **∃-Elimination** rule of natural deduction.

It says that … can be proved from ____, ∃xφ provided that it can be proved from ____, $\varphi[x_0/x]$.

∀R rule

$$\frac{\_\_\_\_ \vdash \varphi[x_0/x], \ldots}{\_\_\_\_ \vdash \forall x\, \varphi, \ldots}\ \vdash \forall R$$

where x_0 is a fresh variable not occurring in ____ or … .

This rule parallels the **∀-Introduction** rule of natural deduction.

It states that to prove ∀xφ it suffices to prove $\varphi[x_0/x]$ where x_0 is an an arbitrary fresh variable.

∃R rule

$$\frac{\_\_\_\_ \vdash \varphi[t/x], \exists x\, \varphi, \ldots}{\_\_\_\_ \vdash \exists x\, \varphi, \ldots}\ \vdash \exists R$$

where t is an term free for x in φ.

This rule parallels the **∃-Introduction rule** of natural deduction.

It states that to prove ∃xφ it suffices to prove φ[t/x] where t is any term.

Sequent Calculus Quantifier Rule Summary

	Left	Right
∃	$\dfrac{\_\_\_, \varphi[x_0/x] \vdash \ldots}{\_\_\_, \exists x\, \varphi \vdash \ldots}\ \exists L$	$\dfrac{\_\_\_ \vdash \varphi[t/x], \exists x\, \varphi, \ldots}{\_\_\_ \vdash \exists x\, \varphi, \ldots} \vdash \exists R$
∀	$\dfrac{\_\_\_, \forall x\, \varphi, \varphi[t/x] \vdash \ldots}{\_\_\_, \forall x\, \varphi \vdash \ldots}\ \forall L$	$\dfrac{\_\_\_ \vdash \varphi[x_0/x], \ldots}{\_\_\_ \vdash \forall x\, \varphi, \ldots} \vdash \forall R$

Sequent Calculus Example

$$\exists x\ \forall y\ p(x, y) \vdash \forall y \exists x\ p(x, y)$$

Sequent Calculus Example

$$\dfrac{\_\_\_, \varphi[x_0/x] \vdash \ldots}{\_\_\_, \exists x\, \varphi \vdash \ldots}\ \exists L$$

$$\exists x\ \forall y\ p(x, y) \vdash \forall y \exists x\ p(x, y)$$

Sequent Calculus Example

$$\dfrac{\forall y\ p(a, y) \vdash \forall y \exists x\ p(x, y)}{\exists x\ \forall y\ p(x, y) \vdash \forall y \exists x\ p(x, y)}\ \exists L$$

Sequent Calculus Example

$$\dfrac{\_\_\_ \vdash \varphi[x_0/x], \ldots}{\_\_\_ \vdash \forall x\, \varphi, \ldots} \vdash \forall R$$

$$\dfrac{\forall y\ p(a, y) \vdash \forall y \exists x\ p(x, y)}{\exists x\ \forall y\ p(x, y) \vdash \forall y \exists x\ p(x, y)}\ \exists L$$

Sequent Calculus Example

$$\dfrac{\dfrac{\forall y\ p(a, y) \vdash \exists x\ p(x, b)}{\forall y\ p(a, y) \vdash \forall y \exists x\ p(x, y)}\ \forall R}{\exists x\ \forall y\ p(x, y) \vdash \forall y \exists x\ p(x, y)}\ \exists L$$

Sequent Calculus Example

$$\frac{\_\_\_,\ \forall x\,\varphi,\ \varphi[t/x]\ \vdash \ldots}{\_\_\_,\ \forall x\,\varphi\ \vdash \ldots}\ \forall L$$

$$\begin{array}{rl}
\underline{\forall y\ p(a, y) \vdash \quad \exists x\ p(x, b)} & \\
\underline{\forall y\ p(a, y) \vdash \forall y \exists x\ p(x, y)} & \forall R \\
\exists x\ \forall y\ p(x, y) \vdash \forall y \exists x\ p(x, y) & \exists L
\end{array}$$

Sequent Calculus Example

$$\begin{array}{rl}
\underline{\forall y\ p(a, y),\ p(a, b) \vdash \quad \exists x\ p(x, b)} & \\
\underline{\forall y\ p(a, y) \vdash \quad \exists x\ p(x, b)} & \forall L \\
\underline{\forall y\ p(a, y) \vdash \forall y \exists x\ p(x, y)} & \forall R \\
\exists x\ \forall y\ p(x, y) \vdash \forall y \exists x\ p(x, y) & \exists L
\end{array}$$

Sequent Calculus Example

$$\frac{\_\_\_\ \vdash \varphi[t/x], \ldots}{\_\_\_\ \vdash \exists x\,\varphi, \ldots}\ \vdash \exists R$$

$$\begin{array}{rl}
\underline{\forall y\ p(a, y),\ p(a, b) \vdash \quad \exists x\ p(x, b)} & \\
\underline{\forall y\ p(a, y) \vdash \quad \exists x\ p(x, b)} & \forall L \\
\underline{\forall y\ p(a, y) \vdash \forall y \exists x\ p(x, y)} & \forall R \\
\exists x\ \forall y\ p(x, y) \vdash \forall y \exists x\ p(x, y) & \exists L
\end{array}$$

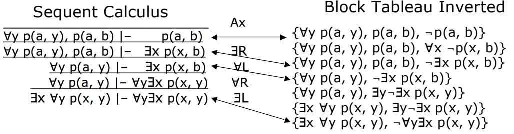
Sequent Calculus Example

$$\begin{array}{rl}
\underline{\forall y\ p(a, y),\ p(a, b) \vdash \quad p(a, b)} & \\
\underline{\forall y\ p(a, y),\ p(a, b) \vdash \quad \exists x\ p(x, b)} & \exists R \\
\underline{\forall y\ p(a, y) \vdash \quad \exists x\ p(x, b)} & \forall L \\
\underline{\forall y\ p(a, y) \vdash \forall y \exists x\ p(x, y)} & \forall R \\
\exists x\ \forall y\ p(x, y) \vdash \forall y \exists x\ p(x, y) & \exists L
\end{array}$$

Sequent Calculus Example

$$\begin{array}{rl}
\overline{} & Ax \\
\underline{\forall y\ p(a, y),\ p(a, b) \vdash \quad p(a, b)} & \\
\underline{\forall y\ p(a, y),\ p(a, b) \vdash \quad \exists x\ p(x, b)} & \exists R \\
\underline{\forall y\ p(a, y) \vdash \quad \exists x\ p(x, b)} & \forall L \\
\underline{\forall y\ p(a, y) \vdash \forall y \exists x\ p(x, y)} & \forall R \\
\exists x\ \forall y\ p(x, y) \vdash \forall y \exists x\ p(x, y) & \exists L
\end{array}$$

Sequent Calculus vs. Tableau



Sequent Calculus

$$\begin{array}{rl}
\overline{} & Ax \\
\underline{\forall y\ p(a, y),\ p(a, b) \vdash \quad p(a, b)} & \\
\underline{\forall y\ p(a, y),\ p(a, b) \vdash \quad \exists x\ p(x, b)} & \exists R \\
\underline{\forall y\ p(a, y) \vdash \quad \exists x\ p(x, b)} & \forall L \\
\underline{\forall y\ p(a, y) \vdash \forall y \exists x\ p(x, y)} & \forall R \\
\exists x\ \forall y\ p(x, y) \vdash \forall y \exists x\ p(x, y) & \exists L
\end{array}$$

Block Tableau Inverted

{∀y p(a, y), p(a, b), ¬p(a, b)}
{∀y p(a, y), p(a, b), ∀x ¬p(x, b)}
{∀y p(a, y), p(a, b), ¬∃x p(x, b)}
{∀y p(a, y), ¬∃x p(x, b)}
{∀y p(a, y), ∃y¬∃x p(x, y)}
{∃x ∀y p(x, y), ∃y¬∃x p(x, y)}
{∃x ∀y p(x, y), ¬∀y∃x p(x, y)}

Block Tableau

{∃x ∀y p(x, y), ¬∀y∃x p(x, y)}
{∃x ∀y p(x, y), ∃y¬∃x p(x, y)}
{∀y p(a, y), ∃y¬∃x p(x, y)}
{∀y p(a, y), ¬∃x p(x, b)}
{∀y p(a, y), p(a, b), ¬∃x p(x, b)}
{∀y p(a, y), p(a, b), ∀x ¬p(x, b)}
{∀y p(a, y), p(a, b), ¬p(a, b)}

Tableau

1.	∃x ∀y p(x, y)	✓	LHS
2.	¬∀y∃x p(x, y)	✓	RHS negated
3.	∃y¬∃x p(x, y)	✓	2, ¬∀ rule
4.	∀y p(a, y)		1, ∃ rule
5.	¬∃x p(x, b)	✓	3, ∃ rule
6.	p(a, b)		4, ∀ rule
7.	∀x ¬p(x, b)		5, ¬∃ rule
8.	¬p(a, b)		7, ∀ rule
	X closes (6, 8)		

Rule Correspondence

Tableau Rule (going downward)	Sequent Calculus Rule (going backward/upward)
¬∃ replace with ∀¬	∃R specializes term on right
¬∀ replace with ∃¬	∀R specializes term on right to fresh var
∃ introduces constant	∃L specializes term on left to fresh var
∀ uses term	∀L specializes term on left
closing path	axiom

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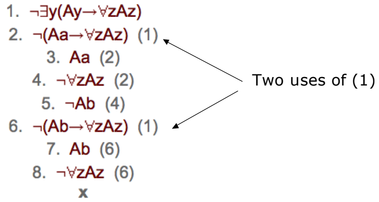
http://www.umsu.de/logik/trees/

Proof Example: $\vdash \exists y\ (A(y) \rightarrow \forall z\ A(z))$



1. $\neg\exists y(Ay \rightarrow \forall zAz)$
2. $\neg(Aa \rightarrow \forall zAz)$ (1)
3. Aa (2)
4. $\neg\forall zAz$ (2)
5. $\neg Ab$ (4)
6. $\neg(Ab \rightarrow \forall zAz)$ (1)
7. Ab (6)
8. $\neg\forall zAz$ (6)

x

Two uses of (1)