Regular Languages

March 23, 2011

CS 81: Computability and Logic

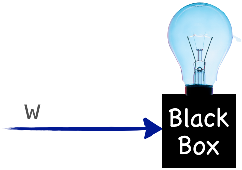
RECALL

1. Questions about computability can be reduced to *decision problems*.
2. Every decision problem can be rephrased as a question about membership in a *language*.

Determining What's Computable

The Plan:

- ✓ Define a class of abstract "machines" that accept or reject strings



- ✓ See what languages this class of machines can recognize (i.e., what decision problems it can solve).

Note:

- ✓ Every machine corresponds to a language (its accepted strings)
- ✓ There may be many different machines accepting the same language
- ✓ If there are restrictions on the machines we can build (e.g., finite size), then not every language may have a machine.

Not Exactly the Machines We're Thinking Of

"When the term 'machine' is used in ordinary discourse, it tends to evoke an unattractive picture. It brings to mind a big, heavy, complicated object which is noisy, greasy, and metallic; performs jerky repetitive, and monotonous motions; and has sharp edges that may hurt one if he does not maintain sufficient distance…"

Marvin Minsky, *Computation: Finite and Infinite Machines*

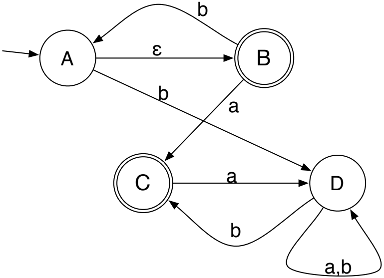
Our First Class of Machines: State Machines

Mathematically, a **state machine** consists of:

1. an alphabet Σ
2. a collection of states Q
3. a transition relation $\rightarrow \quad \subseteq \quad Q \times (\Sigma \cup \{\varepsilon\}) \times Q$
   (where $q \xrightarrow{\sigma} q'$ means that (q, σ, q') is in the relation)
4. one initial state $q_0 \in Q$.
5. a set of final/accepting states $F \subseteq Q$.

finite state machine:
Q is finite

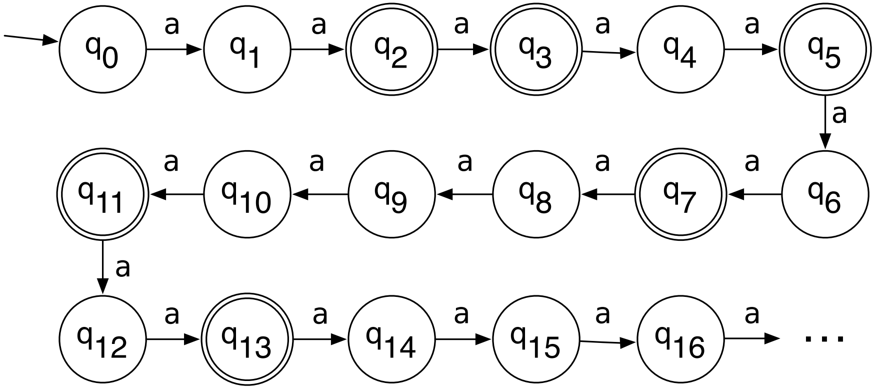
deterministic state machine:
a transition *function* $\delta : Q \times \Sigma \rightarrow Q$.



Machine Behavior

- ✓ The machine starts in state q_0.
- ✓ It can change from state q to state q' on input σ provided that $q \xrightarrow{\sigma} q'$.
- ✓ It can change from state q to state q' spontaneously provided that $q \xrightarrow{\varepsilon} q'$.
- ✓ The machine accepts a string $w \in \Sigma^*$ if there is *at least one* path spelling out w, that starts at q_0 and ends at a state $\in F$

What's Accepted?



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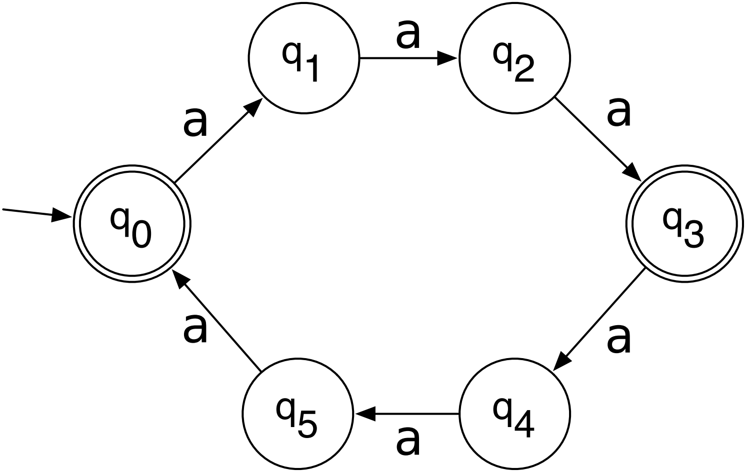
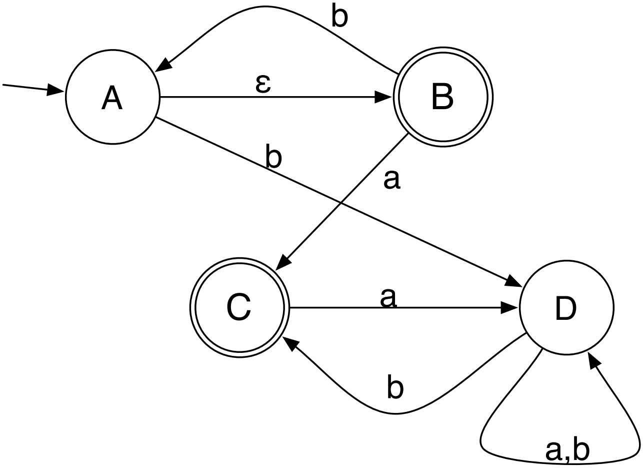


What's Accepted? (bb? b? aaab? ba?)



Finite State Machines

We care mostly about *finite* state machines, also known as "Finite Automata"

Terminology:

- ✓ DFA = Deterministic Finite Automaton = Deterministic FSM
- ✓ NFA = Nondeterministic Finite Automaton = Nondeterministic FSM

A Jewel of Theoretical Computer Science

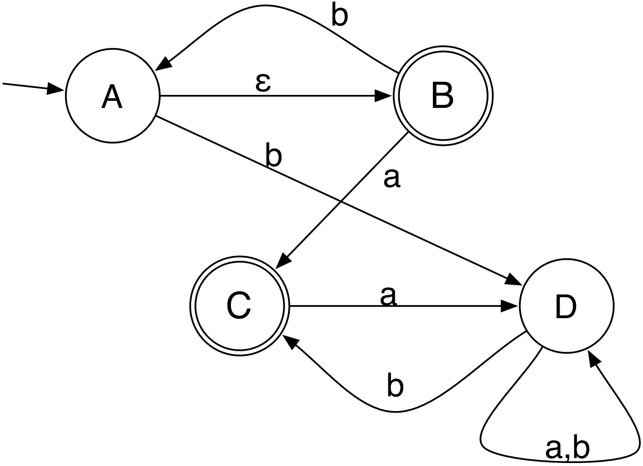


The following are equivalent:

- ✓ There is a DFA accepting the language L
- ✓ [Rabin and Scott] There is an NFA accepting L
- ✓ [Kleene] L is a regular set.

Digression: "Scotland Yard," the game

From NFA to DFA: the **Subset Construction**



Regular Languages

An *inductively-defined* collection of sets!

- ✓ $\emptyset$ is a regular language.
- ✓ $\{a\}$ is regular for any $a \in \Sigma$.
- ✓ If L and M are regular, then so is LM and $L \cup M$.
- ✓ If L is regular, then so is L^*.

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True or False?

1. Σ^* is regular.
2. $\{\varepsilon\}$ is regular.
3. If $w \in \Sigma^*$, then $\{w\}$ is regular.
4. Every finite language is regular.
5. Every set is regular (since $\{w_1, w_2, \ldots\} = \{w_1\} \cup \{w_2\} \cup \cdots$).

Regular Expressions

An *inductively-defined* collection of expressions!

- ✓ $\emptyset$ is a regexp
- ✓ ε is a regexp
- ✓ a is a regexp for any $a \in \Sigma$.
- ✓ If r_1 and r_2 are regexps, then so is $(r_1 r_2)$ and $(r_1|r_2)$.
- ✓ If r is a regexp, then so is (r^*).

Parenthesis Convention:

$$ab^*|c^* \quad = \quad (a(b^*)) \mid (c^*)$$

Regexp Interpretations

Regular expressions abbreviate regular languages.

- ✓ $L(\emptyset) = \emptyset$
- ✓ $L(\varepsilon) = \{\varepsilon\}$
- ✓ $L(a) = \{a\}$
- ✓ $L(r_1 r_2) = L(r_1)\, L(r_2)$
- ✓ $L(r_1 \mid r_2) = L(r_1) \cup L(r_2)$
- ✓ $L(r^*) = L(r)^*$

We say that "r matches w" if $w \in L(r)$. True or False?

- ✓ $L(r_1) = L(r_2) \quad \longrightarrow \quad r_1 = r_2$
- ✓ There is a regular expression r with $L(r) = \Sigma^*$

Regular Expression Examples ($\Sigma = \{0, 1\}$)

Describe the Language

1. $0 \mid 1$
2. $(0|1)^*$
3. $(0|1)\, 0^*\, 1^*$
4. $0^*110^*|1^*001^*$

Find the regular expression

1. Strings where every 1 is followed by a 0.
2. Strings where no 1 is followed by a 0.
3. Strings where every 1 is preceded by and followed by 0.

From Regular Expression to NFA

Construct $\mathrm{NFA}(r)$ by induction/recursion on the regular expression r.

- ✓ $\emptyset$ is a regexp
- ✓ ε is a regexp
- ✓ a is a regexp for any $a \in \Sigma$.
- ✓ If r_1 and r_2 are regexps, then so is $(r_1 r_2)$ and $(r_1 | r_2)$.
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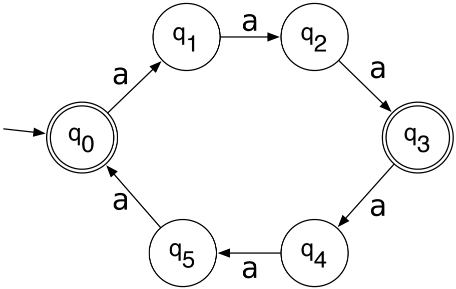
Completing the Equivalence: Automata to Regular Expressions

Two approaches:

1. Solving equations
2. Generalized NFAs

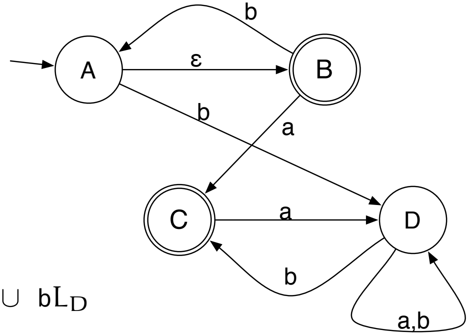
The Language of a State



Let L_q be the set of strings are accepted when starting from state q.

- ✓ What is $L_{q_0}, L_{q_1}, L_{q_2}, \ldots$?
- ✓ How is L_{q_1} related to L_{q_2}?

Automaton as a System of Equations



- $L_A = \varepsilon L_B \ \cup \ bL_D$
- $L_B =$
- $L_C =$
- $L_D =$

Solving Equations using Arden's Rule

- The equation

$$L = AL \cup B$$

has the solution

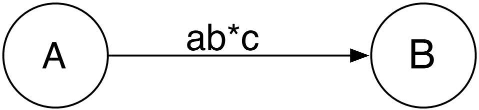
$$L = A^*B$$

- This is the smallest solution
  - If $\varepsilon \notin A$, the unique solution
  - Otherwise A^*C is a solution for any $B \subseteq C$.

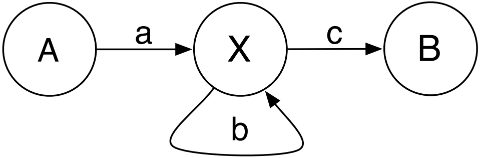
$$\begin{aligned} L_A &= L_B \cup \mathsf{b}L_D \\ L_B &= \varepsilon \cup \mathsf{b}L_A \cup \mathsf{a}L_C \\ L_C &= \varepsilon \cup \mathsf{a}L_D \\ L_D &= (\mathsf{a} \cup \mathsf{b})L_D \cup \mathsf{b}L_C \end{aligned}$$

Generalized NFAs

Just like an NFA, but edges have regular expressions rather than single symbols



Since regular expressions can be turned into NFAs, we aren't adding any extra power.

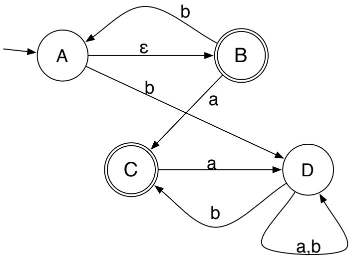


Regexp by Removing States

The strategy:

✓ Make sure our NFA has

- *One start state, with edges only going out*
- *One accept state, with edges only going in.*



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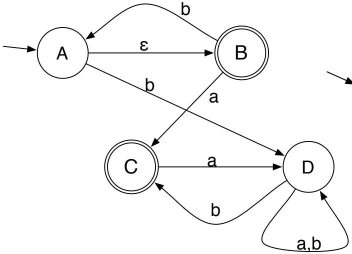

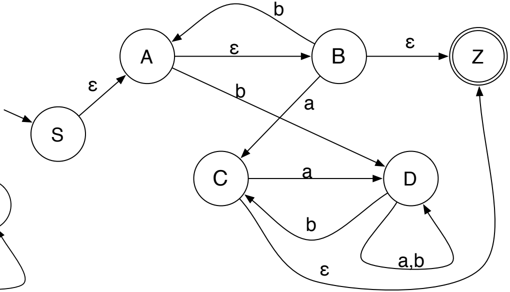

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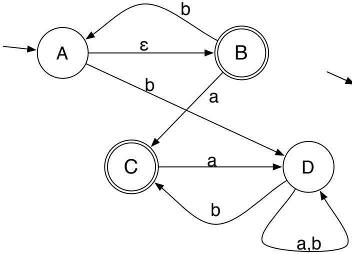
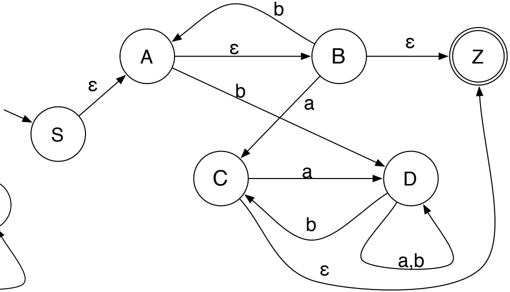
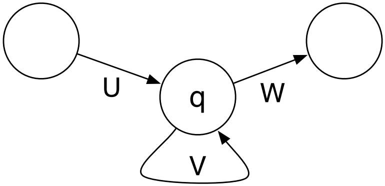




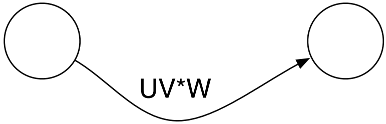
- ✓ Remove all the intermediate states (A–D), one at a time.
- ✓ In the end, we have one edge, labeled by our regexp.

Removing States

✓ When removing state **q**, replace every pair of in/out edges



by a single edge



Example