

Sequent Calculus

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Sequent Calculus (SC)

- The **sequent calculus** was used by Gentzen (he called it “System S”) to derive results about Natural Deduction.
- SC allows **sets** of formulas on **both** sides of \vdash .
- In the SC, \vdash becomes an **object-language** symbol, rather than a meta-language symbol as we have been using it. This is similar to the **contextual representation** used in the soundness proof. However, some authors use \Rightarrow or \rightarrow instead of \vdash .
- It is easier to **implement in software** a proof generator based on SC than it is one based on ND.

Meaning of \vdash in SC

- The *intuitive* meaning of

$$A_1, \dots, A_m \vdash B_1, \dots, B_n$$

“Using hypotheses in the set $\{A_1, \dots, A_m\}$
one can derive **at least one** member of $\{B_1, \dots, B_n\}$ ”.

- If $n = 1$, we have the kind of sequent we've seen before.
- If $n = 0$, it's like the right-hand side is \perp .

Sequent Calculus Rules

- Rather than have a sequence of formulas as in ND, all relevant formulas are represented in the sequent, as in the **contextual form**.
- A proof progresses from one sequent to another.
- It is easiest to start with a goal sequent and work backward (“upward”).
- In some cases, more than one sequent is required to prove a lower sequent.

Convention

- In what follows, Γ and Δ stand for **sets of formulas**.
- Often these sets are shown as Γ and Δ , but our notation seems less cluttered.
- Also Γ, F, G means $\Gamma \cup \{F, G\}$.

Rules for \wedge

- Recall the natural deduction (ND) rules $\wedge E$:

$$\frac{F \wedge G}{F} \qquad \frac{F \wedge G}{G}$$

- In SC, the $\wedge E$ rule corresponds to $\wedge L$ (L for “left”):

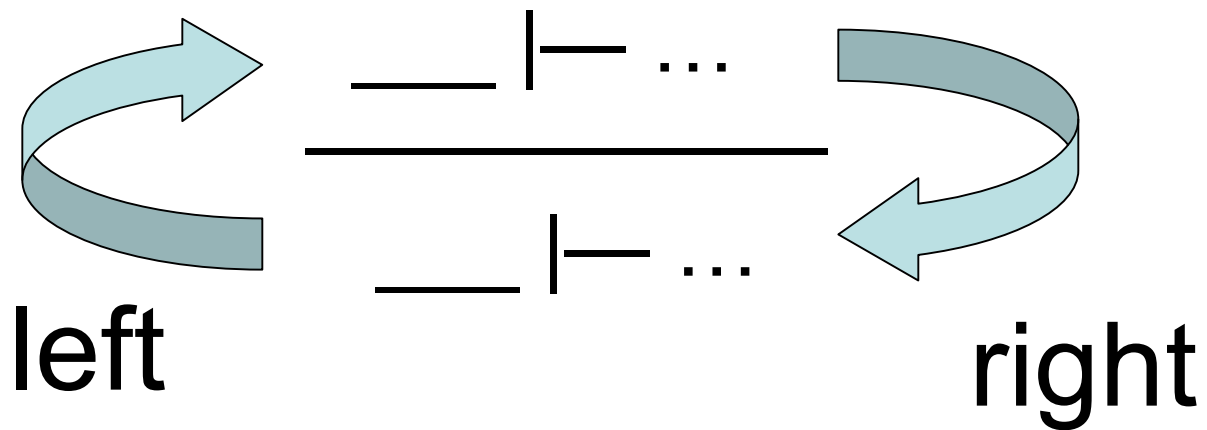
$$\frac{\text{___}, F, G \vdash \dots}{\text{___}, F \wedge G \vdash \dots} \quad \wedge L$$

Meaning of $\wedge L$

$$\frac{\text{____}, F, G \vdash \dots}{\text{____}, F \wedge G \vdash \dots} \quad \wedge L$$

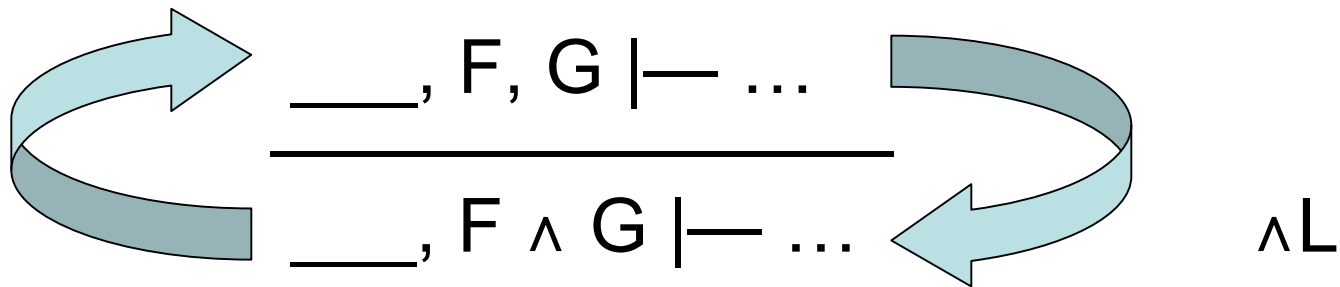
Meaning: If we can deduce ... from ____, F, G
then we can also deduce ... from ____, F \wedge G.

“Information Flow” in an SC Proof



assuming the SC proof is
constructed bottom to top

Information Flow in $\wedge L$



From $F \wedge G$ we can derive both F and G (in ND).

From $\text{___, } F, G$ we can derive \dots .

Rules for $\wedge R$

- Recall the natural deduction (ND) rule $\wedge I$:

$$\frac{F \quad G}{F \wedge G}$$

In SC, the $\wedge I$ rule corresponds to

$$\frac{\frac{}{\quad} \vdash F, \dots \quad \frac{}{\quad} \vdash G, \dots \quad (2 \text{ sequents})}{\frac{}{\quad} \vdash F \wedge G, \dots} \quad \wedge R$$

Here there two sequents above the line, and **each** must be proved. The information flow combines F and G. Bottom up, one sequent “splits” into two.

\wedge Rule Summary

$$\frac{\text{___}, A, B \vdash \dots}{\text{___}, A \wedge B \vdash \dots} \quad \wedge L$$

“To **use** $A \wedge B$, you can use both A and B .”

$$\frac{\text{___} \vdash A, \dots \quad \text{___} \vdash B, \dots}{\text{___} \vdash A \wedge B, \dots} \quad \wedge R$$

“To **derive** $A \wedge B$, derive each separately.”

SC “Axiom”

- In SC, there is one axiom (rule with no antecedent):

$$\frac{}{_, P \vdash P, \dots} \quad \text{Axiom}$$

- Here $_$ and \dots represent sets of formulas as usual.
- The meaning in this case is that any formula can be derived from itself.
- A full derivation tree will have an **axiom at each leaf**.

\vee Rule Summary

$$\frac{\text{___} \vdash A, B, \dots}{\text{___} \vdash A \vee B, \dots} \vee R$$

“To **derive** $A \vee B$, it suffices to derive either.”

(like \vee Introduction in ND)

$$\frac{\text{___}, A \vdash \dots \quad \text{___}, B \vdash \dots}{\text{___}, A \vee B \vdash \dots} \vee L$$

“To **use** $A \vee B$, we need separate derivations **using** A and B .”

(similar to \vee Elimination in ND).

\rightarrow Rules

$$\frac{\text{___}, A \vdash B, \dots}{\text{___} \vdash A \rightarrow B, \dots} \rightarrow R$$

“To **derive** $A \rightarrow B$, it suffices to derive B with A as an added assumption.” (similar to \rightarrow Introduction in ND).

$$\frac{\text{___} \vdash A, \dots \quad \text{___}, B \vdash \dots}{\text{___}, A \rightarrow B \vdash \dots} \rightarrow L$$

“If A can be proved, and we can prove \dots based on B as an added assumption, then \dots can be proved from $A \rightarrow B$.”

(similar to \rightarrow Elimination in ND).

\neg Rules

$$\frac{\text{___}, A \vdash \dots}{\text{___} \vdash \neg A, \dots} \neg R$$

“To **derive** $\neg A, \dots$ from ___ , it suffices to **use** $\text{___}, A$ to get \dots .”
(similar to \neg Introduction in ND)

$$\frac{\text{___} \vdash A, \dots}{\text{___}, \neg A \vdash \dots} \neg L$$

“To **use** $\text{___}, \neg A$ to derive \dots it suffices to derive A, \dots from ___ .”
(similar to RAA in ND)

A way to remember \rightarrow Rules

- Recall that $F \rightarrow G$ is like $\neg F \vee G$.
- So the effect of the \rightarrow can be achieved (bottom up) by using the appropriate \vee rule followed by a \neg -rule.
- In particular, $\rightarrow L$ splits as does $\vee L$. But once the split is done, the F in $\neg F$ can be “flipped” to the other side.

SC Rule Summary

	Introduce on Left	Introduce on Right
\wedge	$\frac{\text{___}, A, B \vdash \dots}{\text{___}, A \wedge B \vdash \dots}$	$\frac{\text{___} \vdash A, \dots \quad \text{___} \vdash B, \dots}{\text{___} \vdash A \wedge B, \dots}$
\vee	$\frac{\text{___}, A \vdash \dots \quad \text{___}, B \vdash \dots}{\text{___}, A \vee B \vdash \dots}$	$\frac{\text{___} \vdash A, B, \dots}{\text{___} \vdash A \vee B, \dots}$
\rightarrow	$\frac{\text{___} \vdash A, \dots \quad \text{___}, B \vdash \dots}{\text{___}, A \rightarrow B \vdash \dots}$	$\frac{\text{___}, A \vdash B, \dots}{\text{___} \vdash A \rightarrow B, \dots}$
\neg	$\frac{\text{___} \vdash A, \dots}{\text{___}, \neg A \vdash \dots}$	$\frac{\text{___}, A \vdash \dots}{\text{___} \vdash \neg A, \dots}$

Assuming “introduce” means from top down, even though we are typically working bottom up.

Sequent Calculus Strategy Summary

Working **backward/upward** from the desired sequent:

Situation	Action
A on left and right	Axiom.
$\neg A$ in a right formula	Flip A to the left.
$\neg A$ in a left formula	Flip A to the right.
$A \wedge B$ in a right formula	Split right set into A and B versions.
$A \wedge B$ in a left formula	Replace the formula with A, B.
$A \vee B$ in a right formula	Replace the formula with A, B.
$A \vee B$ in a left formula	Split left set into A and B versions.
$A \rightarrow B$ in a right formula	Replace with A on the left, B on the right.
$A \rightarrow B$ in a left formula	Split into two versions with A on the right in one, B on the left in the other.

Example Sequent Calculus Proof

Constructed from bottom to top:

$$P \vee Q, \neg P \mid\text{-} Q$$

Example Sequent Calculus Proof

Constructed from bottom to top:

$$\frac{P \vee Q \mid \neg P, Q}{P \vee Q, \neg P \mid \neg Q} \quad \neg\text{-L rule}$$

Example Sequent Calculus Proof

Constructed from bottom to top:

$$\frac{\frac{P \mid\!-\! P, Q \quad Q, \mid\!-\! P, Q}{P \vee Q \mid\!-\! P, Q}}{P \vee Q, \neg P \mid\!-\! Q} \quad \begin{array}{l} \vee\text{L rule} \\ \neg\text{L rule} \end{array}$$

Rough Correspondence: SC vs. ND

- The last sequent of an SC derivation will always correspond to the overall sequent derived in ND.
- Other sequents may correspond to subproofs.
- Moving from **bottom to top** in an SC proof is like working **outside-in** in a ND proof.
- Consider LHS of a sequent to be *all* operative hypotheses, including assumptions in sub-proofs.
- Consider RHS to be goal (as a disjunction).

Conversion Between SC and ND

- http://twelf.plparty.org/wiki/POPL_Tutorial/Sequent_vs_Natural_Deduction
- <http://www.ags.uni-sb.de/~chris/papers/2002-pisa.pdf>

Compare JAPE SC Version (MCS)

$$\begin{array}{c}
 \frac{}{P \mid \neg P, Q} \text{Axiom} \qquad \frac{}{Q, \mid \neg P, Q} \text{Axiom} \\
 \hline
 \frac{P \vee Q \mid \neg P, Q}{P \vee Q, \neg P \mid \neg Q}
 \end{array}
 \quad
 \begin{array}{c}
 \vee \mid \neg \text{ rule} \\
 \neg \mid \neg \text{ rule}
 \end{array}$$

$$\begin{array}{c}
 \frac{\frac{}{P \vdash P, Q} \text{axiom} \quad \frac{}{Q \vdash P, Q} \text{axiom}}{\vee \vdash} \\
 \frac{P \vee Q \vdash P, Q}{\neg \vdash} \\
 P \vee Q, \neg P \vdash Q
 \end{array}$$

MCS = Multi-Conclusion (classical)
 SCS = Single Conclusion (intuitionistic)

More Sequent Calculus Examples (constructed working backward/upward)

$$\vdash ((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightarrow (P \rightarrow R)$$

More Sequent Calculus Examples

$$\frac{(P \rightarrow Q) \wedge (Q \rightarrow R) \quad |- \quad (P \rightarrow R)}{|- \quad ((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightarrow (P \rightarrow R)} \quad |- \rightarrow$$

More Sequent Calculus Examples

$$\frac{(P \rightarrow Q), (Q \rightarrow R) \quad | \vdash (P \rightarrow R)}{\quad}$$

\wedge |-

$$\frac{(P \rightarrow Q) \wedge (Q \rightarrow R) \quad | \vdash (P \rightarrow R)}{\quad}$$

| - \rightarrow

$$| \vdash ((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightarrow (P \rightarrow R)$$

More Sequent Calculus Examples

$$\frac{(P \rightarrow Q), (Q \rightarrow R), P \quad |- \quad R}{(P \rightarrow Q), (Q \rightarrow R) \quad |- \quad (P \rightarrow R)} \quad \rightarrow \text{-} \rightarrow$$
$$\frac{(P \rightarrow Q), (Q \rightarrow R) \quad |- \quad (P \rightarrow R)}{(P \rightarrow Q) \wedge (Q \rightarrow R) \quad |- \quad (P \rightarrow R)} \quad \wedge \text{-} \rightarrow$$
$$\frac{(P \rightarrow Q) \wedge (Q \rightarrow R) \quad |- \quad (P \rightarrow R)}{|- \quad ((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightarrow (P \rightarrow R)} \quad \rightarrow \text{-} \rightarrow$$

More Sequent Calculus Examples

$$\begin{array}{l}
 \frac{(P \rightarrow Q), R, P \vdash R}{(P \rightarrow Q), P \vdash Q, R} \quad \rightarrow \vdash \\
 \frac{(P \rightarrow Q), (Q \rightarrow R), P \vdash R}{(P \rightarrow Q), (Q \rightarrow R) \vdash (P \rightarrow R)} \quad \wedge \vdash \\
 \frac{(P \rightarrow Q) \wedge (Q \rightarrow R) \vdash (P \rightarrow R)}{\vdash ((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightarrow (P \rightarrow R)} \quad \rightarrow \vdash
 \end{array}$$

More Sequent Calculus Examples

$$\begin{array}{c}
 \frac{}{\text{Ax}} \quad \frac{P \vdash Q, R, P}{Q, P \vdash Q, R} \quad \rightarrow \vdash \\
 \frac{(P \rightarrow Q), R, P \vdash R}{(P \rightarrow Q), P \vdash Q, R} \quad \rightarrow \vdash \\
 \frac{(P \rightarrow Q), (Q \rightarrow R), P \vdash R}{\vdash} \rightarrow \\
 \frac{(P \rightarrow Q), (Q \rightarrow R) \vdash (P \rightarrow R)}{\wedge \vdash} \\
 \frac{(P \rightarrow Q) \wedge (Q \rightarrow R) \vdash (P \rightarrow R)}{\vdash} \rightarrow \\
 \vdash ((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightarrow (P \rightarrow R)
 \end{array}$$

An SC Prover in Prolog

```
% If L |- R, where L and R are lists of formula,
% prove(L, R, Proof) will produce a sequent calculus proof tree.
% Using test(L, R) will display the proof in line-numbered fashion.
% If the sequent is not provable, the prove predicate will fail.

prove(L, R, axiom(L, R)) :- member(F, L), member(F, R).

prove(L, R, notLeft(L, R, Proof)) :-
    select(not(A), L, Rest),
    prove(Rest, [A | R], Proof).

prove(L, R, notRight(L, R, Proof)) :-
    select(not(A), R, Rest),
    prove([A | L], Rest, Proof).

prove(L, R, andLeft(L, R, Proof)) :-
    select(and(A, B), L, Rest),
    prove([A, B | Rest], R, Proof).

prove(L, R, andRight(L, R, ProofL, ProofR)) :-
    select(and(A, B), R, Rest),
    prove(L, [A | Rest], ProofL),
    prove(L, [B | Rest], ProofR).

prove(L, R, orLeft(L, R, ProofL, ProofR)) :-
    select(or(A, B), L, Rest),
    prove([A | Rest], R, ProofL),
    prove([B | Rest], R, ProofR).

prove(L, R, orRight(L, R, ProofL, ProofR)) :-
    select(or(A, B), R, Rest),
    prove(L, [A, B | Rest], ProofL),
    prove(L, [B | Rest], R, ProofR).

prove(L, R, impliesRight(L, R, Proof)) :-
    select(implies(A, B), R, Rest),
    prove([A | L], [B | Rest], Proof).

prove(L, R, impliesLeft(L, R, ProofL, ProofR)) :-
    select(implies(A, B), L, Rest),
    prove(Rest, [A | R], ProofL),
    prove([B | Rest], R, ProofR).
```

Formatted Output of the SC Prover

```
% Propositional Sequent Calculus Prover constructed in Prolog
% author: Robert Keller

/* Example output from test([not(or(a, b))], [and(not(a), not(b))]).
 *
 * Proof of sequent or(not(a),not(b)) |- not(and(a,b)):
 * 1: a, b |- a -- Axiom
 * 2: not(a), a, b |- -- not L, line 1
 * 3: a, b |- b -- Axiom
 * 4: not(b), a, b |- -- not L, line 3
 * 5: a, b, or(not(a),not(b)) |- -- or L, lines 2 and 4
 * 6: and(a,b), or(not(a),not(b)) |- -- and L, line 5
 * 7: or(not(a),not(b)) |- not(and(a,b)) -- not R, line 6
 */
```

We use line numbers to display the tree linearly.

JAPE SCS vs. MCS

- SCS: Single-conclusion:
 Will only prove intuitionistic sequents.
 The RHS is always a single formula.
 For some reason
 $\neg\varphi$ converts to $\varphi \rightarrow \perp$ first.

$$\begin{array}{c}
 \frac{\frac{\overline{\text{hyp}} \quad \overline{\text{hyp}}}{P \vdash P} \quad P, \vdash \perp}{P, \vdash \perp}}{\rightarrow \vdash} \\
 \frac{P, \vdash \perp}{P \rightarrow \perp} \\
 \frac{P, \vdash \perp}{\neg P} \\
 \frac{\neg P}{P \vdash \neg P \rightarrow \perp} \\
 \frac{P \vdash \neg P \rightarrow \perp}{P \vdash \neg \neg P}
 \end{array}$$

This is $\neg\neg$ Introduction.

$\neg\neg$ Elimination cannot be proved intuitionistically.

SC vs. Tableaux

- Tableaux proofs are introduced elsewhere. Skip this for now if they haven't been covered.
- A correspondence can be established between an SC proof and a **block tableau** proof T.
- Tableau is constructed top-down; SC is bottom-up.
- The conclusion of SC is negated in T.
- The premises of SC, if any, are unnegated in T.
- There is a correspondence between rules of the two systems.
- Splitting in SC is like splitting in the block tableau.
- Axioms of SC correspond to path closure.

Sequent Calculus vs. Tableaux

Sequent Calculus	Tableau
Constructed bottom-up.	Constructed top-down.
Proves $A_1, \dots, A_m \vdash B_1, \dots, B_n$ Equivalent to proving ($\neg A_1 \vee \dots \vee \neg A_m \vee B_1 \vee \dots \vee B_n$)	Negated conclusion (typically $m = 0, n = 1$) $\neg(\neg A_1 \vee \dots \vee \neg A_m \vee B_1 \vee \dots \vee B_n)$ $\equiv A_1 \wedge \dots \wedge A_m \wedge \neg B_1 \wedge \dots \wedge \neg B_n$
Formulas on the right	Originally negated formulas
Formulas on the left	Originally un-negated formulas
Axiom $\dots, p, \dots \vdash _, p, _$	Closure $p, \neg p$

Tableau vs. SC Example

Block Tableau
$\{ \neg((p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)) \}$
$\{ (p \rightarrow q), \neg(\neg q \rightarrow \neg p) \}$

SC "upside-down"
$\vdash (p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$
$(p \rightarrow q) \vdash (\neg q \rightarrow \neg p)$

Tableau vs. SC Example

Block Tableau
$\{ \neg((p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)) \}$
$\{ (p \rightarrow q), \neg(\neg q \rightarrow \neg p) \}$
$\{ (p \rightarrow q), \neg q, \neg\neg p \}$

SC “upside-down”
$\vdash (p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$
$(p \rightarrow q) \vdash (\neg q \rightarrow \neg p)$
$\neg q, (p \rightarrow q) \vdash \neg p$

Tableau vs. SC Example

Block Tableau
$\{ \neg((p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)) \}$
$\{ (p \rightarrow q), \neg(\neg q \rightarrow \neg p) \}$
$\{ (p \rightarrow q), \neg q, \neg\neg p \}$
$\{ (p \rightarrow q), \neg q, p \}$

SC “upside-down”
$\vdash (p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$
$(p \rightarrow q) \vdash (\neg q \rightarrow \neg p)$
$\neg q, (p \rightarrow q) \vdash \neg p$
$\neg q, (p \rightarrow q), p \vdash$

Tableau vs. SC Example

Block Tableau
$\{ \neg((p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)) \}$
$\{ (p \rightarrow q), \neg(\neg q \rightarrow \neg p) \}$
$\{ (p \rightarrow q), \neg q, \neg\neg p \}$
$\{ (p \rightarrow q), \neg q, p \}$
$\{ \neg p, \neg q, p \}X \quad \{ q, \neg q, p \}X$

SC “upside-down”
$\vdash (p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$
$(p \rightarrow q) \vdash (\neg q \rightarrow \neg p)$
$\neg q, (p \rightarrow q) \vdash \neg p$
$\neg q, (p \rightarrow q), p \vdash$
$\neg q, q, p \vdash \quad q, p \vdash p(ax)$
$q, p \vdash q (ax)$

What are Blocks?

- A block is simply a set of formulas corresponding to the ***un-checked*** formulas on a **single path** in the tree tableau.
- We start with one block.
- **Stacking** replaces formulas **within** that block.
- **Splitting** splits the block into two.
- **Closure** is when a block contains a formula and its negation.

Block Tableau from Tableau Example

Negated Formula at Root

1. $\neg((p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p))$

Corresponding Block Tableau

$$\{ \neg((p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)) \}$$

1. $\neg((p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)) \checkmark$

2. $(p \rightarrow q)$

3. $\neg(\neg q \rightarrow \neg p)$

Corresponding Block Tableau

$\{ (p \rightarrow q), \neg(\neg q \rightarrow \neg p) \}$

1. $\neg((p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)) \checkmark$

2. $(p \rightarrow q)$

3. $\neg(\neg q \rightarrow \neg p) \checkmark$

4. $\neg q$

5. $\neg\neg p$

Corresponding Block Tableau

$\{ (p \rightarrow q), \neg q, \neg\neg p \}$

1. $\neg((p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)) \checkmark$

2. $(p \rightarrow q)$

3. $\neg(\neg q \rightarrow \neg p) \checkmark$

4. $\neg q$

5. $\neg\neg p \checkmark$

6. p

Corresponding Block Tableau

$\{(p \rightarrow q), \neg q, p\}$

1. $\neg((p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)) \checkmark$

2. $(p \rightarrow q) \checkmark$

3. $\neg(\neg q \rightarrow \neg p) \checkmark$

4. $\neg q$

5. $\neg\neg p \checkmark$

6. p

Split from line 2.

7. $\neg p$

8. q

$X(6, 7)$

$X(4, 8)$

Corresponding Block Tableau Splits

$\{\neg p \neg q, p\}$, $\{q, \neg q, p\}$ both blocks close

One Sequent $\neg(\neg E \wedge \neg F) \vdash E \vee F$ Proved Five Ways

Natural Deduction (tree)

$$\begin{array}{c}
 \frac{\frac{\frac{[\neg(E \vee F)]^1}{\frac{\frac{[E]^2}{E \vee F}}{\perp} \text{ } \neg E}}{\perp} \text{ } \neg E \quad \neg I_2}{\neg(\neg E \wedge \neg F)} \quad \frac{\frac{[\neg(E \vee F)]^1}{\frac{\frac{[F]^3}{E \vee F}}{\perp} \text{ } \neg F}}{\perp} \text{ } \neg F \quad \neg I_3}{\neg E \wedge \neg F} \quad \wedge I}{\neg(\neg E \wedge \neg F) \quad \neg E \wedge \neg F} \quad \neg E}{\perp} \text{ } \perp \quad \text{RAA}_1}{E \vee F}
 \end{array}$$

One Sequent $\neg(\neg E \wedge \neg F) \vdash E \vee F$ Proved Five Ways

Natural Deduction (Fitch diagram)

1	$\neg(\neg E \wedge \neg F)$	premise
2	$\neg(E \vee F)$	assumption
3	E	assumption
4	$E \vee F$	3, $\vee I$
5	\perp	2, 4, $\neg E$
6	$\neg E$	3-5, $\neg I$
7	F	assumption
8	$E \vee F$	7, $\vee I$
9	\perp	2, 8, $\neg E$
10	$\neg F$	7-9, $\neg I$
11	$\neg E \wedge \neg F$	6, 10, $\wedge I$
12	\perp	1, 11, $\neg E$
13	$E \vee F$	2-12, RAA

One Sequent $\neg(\neg E \wedge \neg F) \vdash E \vee F$ Proved Five Ways

Sequent Calculus (read bottom-up)

$$\begin{array}{c}
 \frac{E \vdash E, F}{\vdash E, F, \neg E} \quad \frac{F \vdash E, F}{\vdash E, F, \neg F} \quad \vdash \neg \\
 \hline
 \vdash E, F, (\neg E \wedge \neg F) \quad \vdash \neg \\
 \hline
 \frac{\neg(\neg E \wedge \neg F) \vdash E, F}{\neg(\neg E \wedge \neg F) \vdash E \vee F} \quad \vdash \vee
 \end{array}$$

Here the correspondence with the Natural Deduction proof is not 1-1.

One Sequent $\neg(\neg E \wedge \neg F) \vdash E \vee F$ Proved Five Ways

Block Tableau

$\{ \neg(\neg E \wedge \neg F), \neg(E \vee F) \}$	
$\{ \neg(\neg E \wedge \neg F), \neg E, \neg F \}$	
$\{ \neg\neg E \vee \neg\neg F, \neg E, \neg F \}$	
$\{ \neg\neg E, \neg E, \neg F \}$	$\{ \neg\neg F, \neg E, \neg F \}$
$\{ E, \neg E, \neg F \}$	$\{ F, \neg E, \neg F \}$
X	X

The blocks could close one step earlier,
due to $\neg\neg E$ and $\neg E$, etc.

One Sequent $\neg(\neg E \wedge \neg F) \vdash E \vee F$ Proved Five Ways

Tableau (tree form)

- | | |
|--|--|
| 1. $\neg(\neg E \wedge \neg F)$ | premise \checkmark |
| 2. $\neg(E \vee F)$ | negated conclusion \checkmark |
| 3. $\neg E$ | (2 stack) |
| 4. $\neg F$ | (2 stack) |
| 5. $\neg\neg E$ (1 split) \checkmark | 6. $\neg\neg F$ (1 split) \checkmark |
| 7. E (5 $\neg\neg$) | 8. F (5 $\neg\neg$) |
| X (3, 7) | X (4, 8) |

The paths could close one step earlier,
due to $\neg\neg E$ and $\neg E$, etc.

An Automated Sequent Calculus Prover: Defunct? <http://bach.istc.kobe-u.ac.jp/seqprover/> Please use for *checking*, not doing, homework! **Sequent Prover (seqprover)**

Sequent:
 Output style:

[\[Top page\]](#)

Trying to prove with threshold = 0

$$\begin{array}{c}
 \begin{array}{c}
 \text{----- Ax} \\
 p \text{ --> } q, p, r \\
 \text{-----}
 \end{array}
 \quad
 \begin{array}{c}
 \text{----- Ax} \\
 p, r \text{ --> } p, r \\
 \text{-----}
 \end{array}
 \quad
 \begin{array}{c}
 \text{----- Ax} \\
 p, q \text{ --> } q, r \\
 \text{-----}
 \end{array}
 \quad
 \begin{array}{c}
 \text{----- Ax} \\
 p, q, r \text{ --> } r \\
 \text{-----}
 \end{array} \\
 \text{----- L->} \quad \text{----- L->} \\
 p, q \text{ --> } r \text{ --> } p, r \qquad p, q, q \text{ --> } r \text{ --> } r \\
 \text{----- L->} \\
 p, p \text{ --> } q, q \text{ --> } r \text{ --> } r \\
 \text{----- R->} \\
 p \text{ --> } q, q \text{ --> } r \text{ --> } p \text{ --> } r \\
 \text{----- L/\} \\
 (p \text{ --> } q) \wedge (q \text{ --> } r) \text{ --> } p \text{ --> } r
 \end{array}$$

Proved in 0 msec.

Tableau Prover

<http://www.umsu.de/logik/trees/>

Please use for *checking*, not doing, homework!

Tree Proof Generator v2.06 (2007-11-12) [Help/Background](#)

Prove

$((p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p))$

$((p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p))$ is valid.

1. $\neg((p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p))$
 2. $(p \rightarrow q)$ (1)
 3. $\neg(\neg q \rightarrow \neg p)$ (1)
 4. $\neg q$ (3)
 5. $\neg\neg p$ (3)
-
6. $\neg p$ (2)
x
 7. q (2)
x