Environments and Bindings

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What is a Binding?

• A **binding** is an association between an identifier (or variable) and a value.

• We say the identifier is “bound to” the value.

• Examples:
  
  X is bound to 5
  foo is bound to “bar”
  null is bound to OpenList.nil
What is an Environment?

- An environment is a set of bindings.
Why is this important?

• Environments change in going from one part of an expression to another.

• For the most part, such changes can be accomplished non-destructively.

• Old environments are not destroyed, just modified by layering on new bindings.
How is an environment represented?

- There are multiple ways, but in the Rockit interpreter, we use an association list.

- `((X 5) (Y 10) (Z 15) (X 20))`
  - X is bound to 5
  - Y is bound to 10
  - Z is bound to 15

The remaining X has no effect, as association lists are searched front to back.
Changing Environments

• These forms all have a *temporary* effect on the environment:
  
  lambda
  
  let
  
  let*
  
  letrec
Example 1: let

- Say the ambient environment is
  `[((X 5) (Y 10) (Z 15) (X 20))`

Consider the expression

`(let ((Y 30)) (+ X Y Z))`
Example 1: let

- Say the ambient environment is
  `'((X 5) (Y 10) (Z 15))`

Consider the expression

```
(let ((Y 30)) (+ X Y Z))
```

Evaluate the expression within the body environment:

```
'((Y 30)(X 5) (Y 10) (Z 15))
```

The `Y` binding is overridden by the `Y 30` inside the `let` expression.

Outer environment:

```
'((X 5) (Y 10) (Z 15))
```

Body environment:

```
'((Y 30)(X 5) (Y 10) (Z 15))
```

Overridden binding:

```
'((Y 30) (X 5) (Y 10) (Z 15))
```
Example 2: nested lets

Consider the expression

```lisp
(let ((Y 30)) (let ((Z 0)) (+ X Y Z)))
```

outer environment

```
((X 5) (Y 10) (Z 15))
```
Example 2: nested lets

Consider the expression

(let ((Y 30)) (let ((Z 0)) (+ X Y Z)))
Example 2: nested lets

Consider the expression

(let ((Y 30)) (let ((Z 0)) (+ X Y Z)))

outer environment

innermost body environment

mezzanine body environment

outer environment
Example 3: Variables in right-hand sides of equations

- The variable bindings defined by let expressions can be viewed as determined from “equations”:

  \[
  (\text{let } ((Y 30)) \ldots) \text{ is like } Y = 30
  \]

  The right-hand sides can contain variables

  \[
  (\text{let } ((Y (* Z Z))) \ldots) \text{ is like } Y = (* Z Z)
  \]
Example 3: Variables in right-hand sides of equations

• This raises the question of “In what environment are the right-hand sides evaluated?”

• The answer is: in the environment containing the let expression.
Example 3: nested lets

Consider the expression

(let ((Y (+ X 2))) (let ((Z (+ X Y))) Z))

'(((X 5) (Y 10) (Z 15))

outer environment
Example 3: nested lets

Consider the expression

(let ((Y (+ X 2))) (let ((Z (+ X Y))) Z))

outer environment

mezzanine body environment

outer environment

mezzanine body environment
Example 3: nested lets

Consider the expression

\[(\text{let } ((Y \ (+\ X\ 2)))\ (\text{let } ((Z\ (+\ X\ Y)))\ Z))\]

\[
\begin{align*}
\text{outer environment} & \quad \text{mezzanine body environment} \\
\text{innermost body environment} & \quad \text{outer environment}
\end{align*}
\]
Example 4: “Redefining” lets

- (let ((X (* X X))) X)

`(X 5)

inner environment

`(X 25) (X 5))

outer environment
Layering

• When interpreting a let expression, it is not necessary to change the outer environment.

• Instead, just lay on new bindings for evaluating the body of the let.

• When evaluation is complete, the original environment will still be intact.
Let* vs. Let

• let* is like a cascade of nested lets:

\[
\text{(let* ((X 5) (Y (* X X)) (X (+ X Y))) ...)}
\]

is equivalent to

\[
\text{(let ((X 5))}
\text{  (let ((Y (* X X)))}
\text{    (let ((X (+ X Y)))}
\text{      ...)
\text{  )
\text{)}
\text{)
\text{)}
\]
Letrec

• In letrec, the environment for the RHS is the same as the inner environment.

• (letrec ((f (lambda(x)
   (if (< x 2) 1 (* x (f (- x 1)))))))
   (f 5)
)
works, but will not if letrec is replaced with let. Why?
letrec vs. let

> (letrec ((f (lambda(x) (if (< x 2) 1 (* x (f (- x 1))))))) (f 10))
3628800

> (let ((f (lambda(x) (if (< x 2) 1 (* x (f (- x 1))))))) (f 10))
.. f: undefined;
cannot reference an identifier before its definition
Using `define`

- Will this work?

```lisp
> (define f (lambda(x) (if (< x 2) 1 (* x (f (- x 1))))))

> (f 10)
```
The base environment

• *define*, at the top level, adds bindings to the base environment.

• The base environment *is used if no binding exists otherwise*.

> (define f (lambda(x) (if (< x 2) 1 (* x (f (- x 1))))))
> (f 10)
3628800
In Rockit Interpreter

- In the variable lookup part of eval, have it check the base environment after checking the environment passed to eval.
public static Object eval(Object ob, OpenList env)
{
    if( ob instanceof OpenList )
    {
        // An OpenList expression is some kind of form
        return evalForm((OpenList)ob, env);
    }

    if( ob instanceof String )
    {
        // A String is a variable
        OpenList found = env.assoc(ob);

        if( found.nonEmpty() )
        {
            return found.second();
        }

        // Checking the base environment after the environment arg enables
        // recursion in a natural way.
        found = baseEnvironment.assoc(ob);

        if( found.nonEmpty() )
        {
            return found.second();
        }
    }
Lambda Expressions

• Lambda expressions translate into function objects (usually called “closures”) that carry their environment with them.

• Sometimes this matters, sometimes not (when?)

• The base environment is still checked if a binding is not found.
Bindings for Lambda Expressions

\[(\text{lambda} \ (X) \ (+ \ X \ Y))\]

\[\text{ambient environment (for example)}\]

\[\text{body environment}\]

The body environment is only known when the function is applied.
Bindings for Lambda Expressions

The body environment is only known when the function is applied.
How to Bind Lambdas

If there is a keyword `lambda` immediately inside the opening left paren, the expression represents a function. Otherwise it is an application.

Start from the outermost application that can be evaluated and work inward, adding binding layers as you go.

Bindings can also be represented by replacing free variables with their values.
Constructing/Deconstructing Examples

• (lambda (X) …) function
• (((lambda (X) …) 4) application
• If the above function returns a function, then (((lambda (X) …) 4) 5) is that function applied.

Example: function returning a function

• (((lambda (X) (lambda(Y) (list X Y)) 4) 5) ((lambda(Y) (list 4 Y)) 5)
  (list 4 5)
  [This is the “Curried” version of the list function.]
Similarity is not Identity

(((lambda (X) (lambda(Y) (list X Y)) 4) 5) before vs.
((lambda(X) ((lambda(Y) (list X Y)) 4)) 5) now

In the second case, the body of the outermost function is itself the *result* of an application: 
(((lambda(Y) (list 5 Y)) 4)), but not a function itself.

Gives ‘(5 4), not ‘(4 5) as before.
Tree Representation

- When all else fails, expressions can be parsed as a **tree**.

\[
\text{(lambda (X) ...)} \quad \text{function} \quad \text{(___ ...) application}
\]

\[
\lambda X \quad \text{Applies a function ___ to ....}
\]
Tree Example

- (((\(X\) (\(Y\) (list \(X\) \(Y\))) \(4\)) \(5\))

application, or function?
Tree Example

• (((lambda (X) (lambda(Y) (list X Y)) 4) 5)

(application, or lambda exp?)
Tree Example

• \(((\lambda (X) (\lambda (Y) \text{list } X \ Y)) \ 4) \ 5)\)

application, or lambda exp?
Tree Example

- (((\(X\)) (\(\lambda Y\) (\(list\ X\ Y\))) 4) 5)
Evaluating

• Application annihilates lambda

\[ \lambda X \ldots \]

Substitute \ldots for all free occurrences of \( X \) in \( \ldots \).

[This step is called \( \beta \) reduction in CS & Logic.]
Evaluating

\[
\lambda X (\text{list } X Y) \rightarrow_\beta (\text{list } 4 Y)
\]
Evaluating

\( \lambda Y \) sub for \( Y \)

(list 4 \( Y \))

\( \beta \)

(list 4 5)
Example

• \(((\lambda(X) \ ((\lambda(Y) \ (\text{list } X \ Y)) \ 4)) \ 5)\)

• Construct tree, then evaluate.
Example

• \(((\text{lambda}(X) \ (\text{lambda}(Y) \ (\text{list} \ X \ Y)) \ 4)) \ 5)\)

Application or function?
Example

• ((lambda(X) ((lambda(Y) (list X Y)) 4)) 5)

(lambda(X) ((lambda(Y) (list X Y)) 4))
Example

• \(((\lambda(X) ((\lambda(Y) (\text{list } X Y)) \ 4)) \ 5)\)
Example

- $$(((\lambda(X) ((\lambda(Y) (\text{list } X Y)) 4)) 5)$$
Example

• \(((\text{lambda}(X) \ ((\text{lambda}(Y) \ (\text{list} \ X \ Y)) \ 4)) \ 5)\)
Two Trees Compared

\[
\lambda X \\text{ (list } X Y) \quad \lambda X \\text{ (list } X Y)
\]

\[
\lambda Y \quad \lambda Y
\]

5

4

4

5
Example

- \(((\text{lambda}(X) ((\text{lambda}(Y) (\text{list } X \ Y)) \ 4)) \ 5)\)
Example

- \(((\lambda(X) ((\lambda(Y) (\text{list } X \ Y)) \ 4)) \ 5)\)

\[\begin{align*}
\lambda Y & \quad 4 \\
\text{sub for } Y & \\
(\text{list } 5 \ Y) & \quad \text{beta} & \quad (\text{list } 5 \ 4) \\
& \quad \text{vs.} \\
& \quad (\text{list } 4 \ 5) \\
& \quad \text{for other tree}
\end{align*}\]
Static vs. Dynamic Binding

• Mathematical functions demand static binding

```
(define y 99)
(define f (lambda(x) (+ x y)))
(let ((y 1000)) (f 1))
```

y in (+ x y) should be bound to 99 (static binding)

If it were dynamic binding, y would be bound to 1000 in (+ x y).
(let ((X 4))
  (lambda(Y) (list X Y)) 5)
)

The application applies a function
  (lambda(Y) (list X Y)),
with the static binding (X 4).

This function applied to 5 gives `(4 5).
Same deal: The application applies a function
\((\lambda(Y) \ (\text{list } X \ Y)))\),
with the static binding \((X \ 4)\).

This function applied to 5 gives \('(4 \ 5)'\).
Static Binding Example 3

(let ((X 4))
  (let ((f (lambda(Y) (list X Y))))
    (let ((X 99))
      (f 5)
    ))
)

Still same deal: The application applies a function
  (lambda(Y) (list X Y)),
  with the static binding (X 4).

This function applied to 5 gives '(4 5).
Dynamic Binding Example 1

(let ((X 4))
  (lambda(Y) (list X Y)) 5)
)

The application applies a function \textit{expression} (lambda(Y) (list X Y))).

X is bound to 4 in the environment of the application, so that is used.

This function applied to 5 gives `(4 5).
Dynamic Binding Example 2

(let ((X 4))
  (let ((f (lambda(Y) (list X Y))))
    (f 5))
)

Same deal: The application applies a function expression
  (lambda(Y) (list X Y))
X is bound to 4 in the environment of the application, so that is used.
This function applied to 5 gives ‘(4 5).
Dynamic Binding Example 3

(let ((X 4))
  (let ((f (lambda(Y) (list X Y))))
    (let ((X 99))
      (f 5))
  )
)

Different: The application is a function expression.

(lambda(Y) (list X Y)),

The application environment has X bound to 99.

The result is `(99 5), not `(4 5) the mathematical result.
Quasi-Static bindings

- In Racket, **bindings** in the **base environment** are statically bound to **locations (variables)** rather than values.

```racket
> (define x 99)
> (define g (lambda(y) (+ x y)))
> (g 1)
100

> (set! x 100) ; destructively modifies the x value
> (g 1)
101 ; The location hasn’t changed,
; but the value has.
```
Let as Lambda

• Lambda is more general than Let

• \[(\text{let } ((V1 \ E1) \ (V2 \ E2) \ \ldots \ (Vn \ En)) \ E0)\]

= \n
\[(((\lambda (V1 \ V2 \ \ldots \ Vn) \ E0) \ E1 \ E2 \ \ldots \ En)\]

However, the static binding issue does not arise with let.
Double Example

• (define double (lambda (f) (lambda (x) (f (f x)))))

• Draw the tree

• Evaluate ((double square) 5)

• Evaluate (((double double) square) 5)
(double double)

> (double double)
#<procedure>

> (((double double) square) 5)
152587890625

> (square (square (square (square 5)))))
152587890625
How Big?

(((double (double double)) square) 5)
How Big?

shot from my screen at 7-point font:

> (/ (log (((double (double double)) square) 5)) (log 10))
45,807 ; decimal digits
Composing Functions Using Functions

```scheme
> (define (compose f g) (lambda (x) (f (g x))))

> (define (cube x) (* x x x))
> ((compose cube square) 5)
  15625

> ((compose square cube) 5)
  15625

> (define (add2 n) (+ 2 n))
> ((compose add2 square) 5)
  27

> ((compose square add2) 5)
  49
```

Draw a picture