# Assignment 2: Propositional Logic

Due: 11:00am, Wednesday, September 19

- Emails about this assignment should be directed to `cs81help@cs.hmc.edu`.
- The usual collaboration rules apply. You may *discuss* an exercise with any other student(s) currently taking CS 81 as long as:
  - You contribute equally;
  - You come away from this discussion only with *understanding in your head* — no written materials or computer notes may be retained;
  - Your submission is authored solely by you, on a separate occasion.
- You should refer only to materials from this semester of CS 81 (lecture notes, handouts, textbooks, grutors, profs, etc.).
- Bring a writeup/printout to class. Illegible answers will get no credit. (For this reason, the grutors recommend LaTeX. But it's up to you.)
- The assignments web page has links to some resources for formatting your proofs.
- Make sure your submission includes your name!

## Reading

In Huth & Ryan, review Chapter 1 *up to and including* Section 1.4.1 (except for Section 1.3, which you are free to skip). **Pay particular attention to the example proofs in Section 1.2, since they'll get you in the right mindset to do proofs of your own!**

There are further hints on doing proofs on the last page of this handout.

## Exercises

1. Give proofs for the following propositions from Exercise 1.2 (pages 79–80) of Huth & Ryan: 1(p), 1(s), 3(c), 3(e), 3(l).

2. **Pierce's Law** is the formula

$$((p \to q) \to p) \to p$$

   (a) By using truth tables, show that Pierce's law is a tautology, (i.e., that

$$\models ((p \to q) \to p) \to p,$$

   or equivalently that this formula is true under every valuation).

   (b) Prove Pierce's Law.

   > **Hint**: *As mentioned in class (and in Section 1.2.5 of Huth & Ryan),* constructive *propositional logic rejects the use of "indirect" arguments. Consequently, in constructive logic one cannot use the excluded middle (*$\vdash p \lor \neg p$*), double-negation elimination (*$\neg\neg p \vdash p$*), or proof by contradiction (if* $\neg p \vdash \bot$*, then* $\vdash p$*) for an arbitrary proposition* $p$*. It turns out that in constructive logic, Pierce's Law is not provable. Therefore, your proof has to involve at least one use of a classical rule.*

3. **Proof or No Proof?** For each of the following, either show there is no proof (by finding a valuation/model that makes the assumptions true and the conclusion false), or provide a natural deduction proof.

   (a) Premise: $\neg p \lor (q \to p)$. Conclusion: $\neg p \land q$.

   (b) Premise: $p \to q$ and $\neg p \to q$. Conclusion: $q$.

   (c) Premise: $p \to q \to r$. Conclusion: $q \to p \to r$.

   (d) Premise: $p \to q \to r$. Conclusion: $p \to r \to q$.

   (e) Premise: $(p \to q) \to r$. Conclusion: $p \to q \to r$.

   (f) Premise: $p \to q$ and $s \to t$. Conclusion: $p \lor s \to q \land t$.

4. **OS Wars**: Consider the following argument:

   - Premise 1: If my computer runs Windows, then Microsoft got my money.
   - Premise 2: If my computer runs Mac OS X, then Apple got my money.
   - Conclusion: At least one of the following statements is true:
     - If my computer runs Mac OS X, then Microsoft got my money
     - If my computer runs Windows, then Apple got my money

   Either provide a formal proof that this argument holds, or find a valuation where the assumptions are true but the conclusion is false. You may want to abbreviate certain propositions (e.g., as $p$, $q$, etc.) for the purposes of your valuation or your proof, but be sure to define any such abbreviations.

5. Recall that

$$
\begin{aligned}
z \notin X &\iff \neg(z \in X) \\
z \in (X \cup Y) &\iff (z \in X) \vee (z \in Y) \\
z \in (X \cap Y) &\iff (z \in X) \wedge (z \in Y) \\
z \in (X \setminus Y) &\iff (z \in X) \wedge \neg(z \in Y)
\end{aligned}
$$

   Suppose we want to prove that

$$
x \in (A \setminus B) \cup (A \cap C) \quad \rightarrow \quad x \in A \setminus (B \setminus C).
$$

   (a) Translate the statement into a propositical-logic statement involving the three predicates $x \in A$, $x \in B$, and $x \in C$.

   (b) Give a natural deduction proof of this statement. Be careful to use only the official natural deduction rules, and not other properties of sets or logic.

   (c) Give a proof that if $x \in (A \setminus B) \cup (A \cap C)$ then $x \in A \setminus (B \setminus C)$ in "mathematical English," in a form that would convince a Mathematics professor.

   (d) How are the two proofs similar? How are the two proofs different?

6. Answer the following survey questions:

   (a) How long did you spend on this assignment?

   (b) What was easiest? What was most difficult?

   (c) With whom, if anyone, did you collaborate with on this assignment?

## Further Hints

- Make sure you understand all the sample proofs in Section 1.2!
- A common question is "What do I do with an *assumption* of the form $\neg A$?". The answer is "It depends."

  However, the most common pattern is to show that other assumptions (either those you were given or those you have assumed inside a box) imply $A$, and then use the fact that you've reached a contradiction.

  [See lines 3–5 in the left half of Example 1.20, or lines 2–4 in the second proof on page 25.]
- Another common question is "How do I prove a *conclusion* of the form $A \vee B$?" The answer is "It depends."

  However, unless you can show that $A$ is definitely true or that $B$ is definitely true, the most common pattern is to use proof by cases (a.k.a. or-elimination). In one case prove $A$ (and hence $A \vee B$ by or-introduction), and the other case prove $B$ (hence $A \vee B$). In either case, $A \vee B$, so we can conclude it must be true.

  Which two cases? Well, if you already have some other disjunction around (assumed or proved) you might do cases on which half is true. Alternatively, you can use the law of the excluded middle to prove your own disjunction $C \vee \neg C$ (for some specific and relevant formula $C$); then do one case where $C$ is assumed, and one case where $\neg C$ is assumed.

  [See Example 1.24.]