Assignment 4: More Predicate Logic

Due: 11:00am, Wednesday, October 3

- Emails about this assignment should be directed to `cs81help@cs.hmc.edu`.
- The usual collaboration rules apply. You may *discuss* an exercise with any other student(s) currently taking CS 81 as long as:
  - You contribute equally;
  - You come away from this discussion only with *understanding in your head* — no written materials or computer notes may be retained;
  - Your submission is authored solely by you, on a separate occasion.
- You should refer only to materials from this semester of CS 81 (lecture notes, handouts, textbooks, grutors, profs, etc.).
- Bring a writeup/printout to class on the due date. Illegible answers will get no credit.
- Make sure your submission includes your name!

Reading

Review Section 2.4 of Huth & Ryan.

1 Tableau Proofs

For each of the following, give a tableau proof.

1. $\vdash \quad (p \to q) \vee (q \to r)$
2. $(p \wedge q) \to r,\ r \to s, q \wedge \neg s \quad \vdash \quad \neg p$
3. $(\exists x.\, P(x)) \to (\exists y. Q(y)) \quad \vdash \quad \exists z.\, (P(z) \to Q(z))$
4. $\forall x.\, \big((P(x) \to Q(x)) \wedge (Q(x) \to P(x))\big) \quad \vdash \quad (\forall x.\, P(x)) \to (\forall x.\, Q(x))$

2 Proof or No Proof?

For each of the following, give a **natural deduction** proof if one exists.

If there is no proof, you should

1. Define a model that makes the assumptions true and the conclusion false;
2. Briefly explain why the assumptions are true and the conclusions are false;
3. *Also*, find an alternate model that makes the conclusion *true*. (It doesn't matter whether this new model makes the assumptions true or not; this is just more practice with models.)

1. Assumption: $\forall x.\, \forall y.\, (S(x,y) \to S(y,x))$
   Conclusion: $\forall x.\, \neg S(x,x)$
2. Assumption: $\top$ (i.e., no assumption)
   Conclusion: $\forall x.\, \forall y. \big(S(x,y) \to \exists w.\, (S(x,w) \wedge S(w,y))\big)$
3. Assumption: $\forall x.\, \big((P(x) \to Q(x)) \wedge (Q(x) \to P(x))\big)$
   Conclusion: $(\forall x.\, P(x)) \to (\forall x.\, Q(x))$
4. Assumption: $(\forall x.\, P(x)) \to (\forall x.\, Q(x))$
   Conclusion: $\forall x.\, \big((P(x) \to Q(x)) \wedge (Q(x) \to P(x))\big)$
5. Assumption: $(\forall x.\, P(x)) \to q$
   Conclusion: $\forall x.\, (P(x) \to q)$
6. Assumptions: $\exists x.P(x)$, $\exists y.Q(y)$
   Conclusion: $\exists z.\, (P(z) \wedge Q(z))$
7. Assumption: $(\exists x.P(x)) \vee (\exists y.Q(y))$
   Conclusion: $\exists z.\, (P(z) \vee Q(z))$
8. Assumption: $\forall x.\, \exists y.\, S(x,y)$
   Conclusion: $\exists y.\, \forall x.\, S(x,y)$

3 Bounded Quantifiers

In class, we mentioned that the "bounded quantifier" notation (where we restrict the individuals being quantified over) can be translate into the more primitive notions on the right.

$$\begin{array}{lcl} \exists x \in S.\, P(x) & \longrightarrow & \exists x.\, (x \in S \wedge P(x)) \\ \forall x \in S.\, P(x) & \longrightarrow & \forall x.\, (x \in S \rightarrow P(x)) \\ \\ \exists x \leqslant n.\, P(x) & \longrightarrow & \exists x.\, (x \leqslant n \wedge P(x)) \\ \forall x \leqslant n.\, P(x) & \longrightarrow & \forall x.\, (x \leqslant n \rightarrow P(x)) \end{array}$$

It might be unexpected that the translations are different for the different quantifiers: bounded-$\exists$ becomes a logical-and, while the translation of bounded-$\forall$ becomes an implication.

1. Describe a model $\mathcal{M}_1$ (where the set of individuals is the set of $\mathbb{N}$ of natural numbers, and the relation $\leqslant$ is interpreted as the actual less-than-or-equal-to relation on $\mathbb{N}$) that makes

   $$\forall x \leqslant n.\, f(x) \leqslant m$$

   and hence

   $$\forall x.\, (x \leqslant n \rightarrow f(x) \leqslant m)$$

   true but makes

   $$\forall x.\, (x \leqslant n \wedge f(x) \leqslant m)$$

   false. (That is, complete the model by giving interpretations of the function f and the constants n and m.)

2. Describe a model $\mathcal{M}_2$ (where the set of individuals is $\mathbb{N}$ and the relation $\leqslant$ is interpreted as less-than-or-equal-to relation on $\mathbb{N}$) that makes

   $$\exists x.\, (x \leqslant n \rightarrow f(x) \leqslant m)$$

   true but makes

   $$\exists x \leqslant n.\, f(x) \leqslant m$$

   and hence

   $$\exists x.\, (x \leqslant n \wedge f(x) \leqslant m)$$

   false.

4 Induction, Revisited

Whenever we define a set inductively, we get an "induction principle" that justifies proof by induction on that set. It says that, if the base cases have some property, and all formation rules preserve the property, then everything in the set satisfies the property.

For example, if we define a set N inductively by saying

- $\mathsf{Z} \in \mathsf{N}$
- If $n \in \mathsf{N}$ then $\mathsf{S}n \in \mathsf{N}$

then the corresponding induction principle is: for any predicate/property P,

$$P(\mathsf{Z}) \quad \wedge \quad (\forall n{\in}\mathsf{N}.\ P(n) \to P(\mathsf{S}n)) \qquad \to \qquad (\forall n{\in}\mathsf{N}.\ P(n)).$$

The resulting set $\mathsf{N} = \{\mathsf{Z}, \mathsf{SZ}, \mathsf{SSZ}, \mathsf{SSSZ}, \ldots\}$ is isomorphic to the set of natural numbers (i.e., Z stands for zero and S stands for successor). It is thus no coincidence that the induction principle for N corresponds exactly to the principle of induction on natural numbers $\mathbb{N}$ familiar from your math classes: for any predicate/property P we have

$$P(0) \quad \wedge \quad (\forall n{\in}\mathbb{N}.\ P(n) \to P(n{+}1)) \qquad \to \qquad (\forall n{\in}\mathbb{N}.\ P(n)).$$

1. Consider the inductive proof you did for Problem 2 in Assignment 1. (If you had significant trouble with it, you can start from the proof in our Sample Solution on Sakai.) It implicitly relies a similar induction principle for counting numbers (whole numbers starting at one, rather than zero):

   $$P(1) \quad \wedge \quad (\forall n{\geqslant}1.\ P(n) \to P(n{+}1)) \qquad \to \qquad (\forall n{\geqslant}1.\ P(n)).$$

   The conclusion of your proof is that $\forall n{\geqslant}1.\ P(n)$ holds for some property $P(n)$. Exactly what property $P(n)$ is this?

2. The principle of induction on counting numbers tells us that in order to prove $\forall n{\geqslant}1.\ P(n)$, it suffices (e.g., by an Modus Ponens implicit in telling the reader that you are using "proof by induction") to prove

   $$P(1) \quad \wedge \quad (\forall n{\geqslant}1.\ P(n) \to P(n{+}1)).$$

   Such a proof requires (at least) uses of $\wedge i$, $\forall i$, and two uses of $\to i$. These natural deduction rules aren't mentioned in your proof for Problem 2, but all the necessary pieces are there. Carefully explain what parts of your proof correspond (explicitly or implicitly) to uses of these three natural deduction steps.

3. In Problem 1 of Assignment 1, the bogus proof that all horses are the same color is structured as a proof by induction, trying to prove that $\forall n{\geqslant}1.\ P(n)$ for a different property $P(n)$. What is $P(n)$ for this problem? [The problem with the proof is *not* the inductive structure, but rather that the subproof of $\forall n{\geqslant}1.\ P(n) \to P(n{+}1)$ contains an error!]