Languages that are Regular and Languages that are not

November 12, 2012

CS 81: Computability and Logic

Closure Properties

A *family* of languages is a set of languages.

- ✓ The family of all finite languages
- ✓ The family of all languages
- ✓ The family of all regular languages

A family $\mathcal{F}$ is *closed under* an operation if applying the operation to languages in $\mathcal{F}$ always produces a result in $\mathcal{F}$.

Finite Languages

Is the family of finite languages closed under:

- ✓ Union? $(A \cup B)$
- ✓ Intersection $(A \cap B)$
- ✓ Concatenation? (AB)
- ✓ Star (A^*)
- ✓ Complement (A^c)

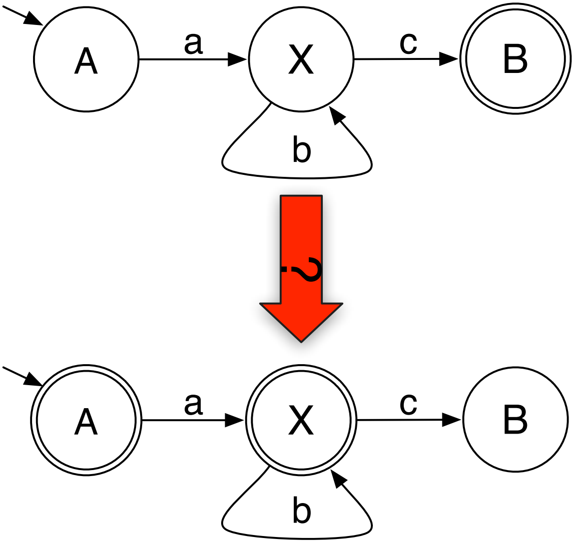
Regular Languages

The regular languages are closed under

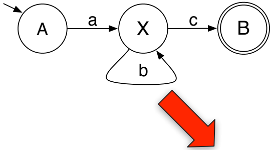
- ✓ Union? $(A \cup B)$
- ✓ Intersection $(A \cap B)$
- ✓ Concatenation? (AB)
- ✓ Star (A^*)
- ✓ Complement (A^c)

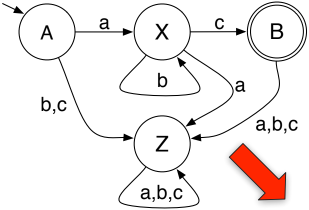
Proofs?

Complement?

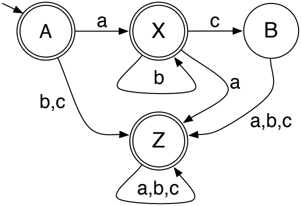


COMPLEMENT!

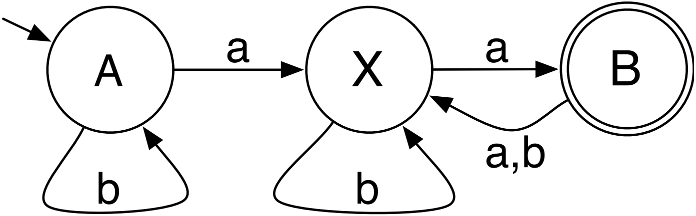


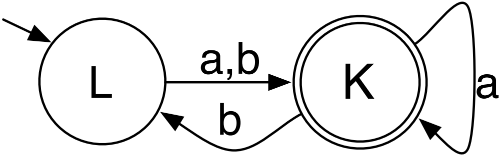


$$\text{DFA } M = (\Sigma, K, \to, q_0, A)$$
$$\text{DFA } M^c = (\Sigma, K, \to, q_0, K \setminus A)$$

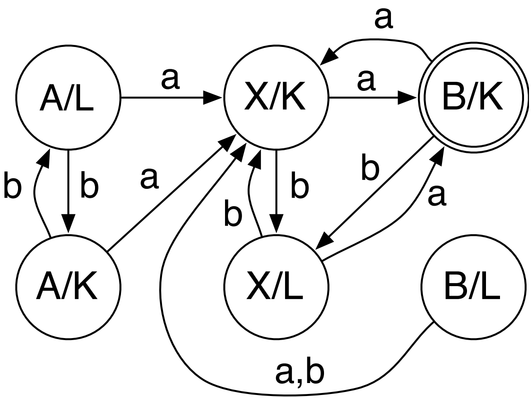


Intersection: DFA Inputs





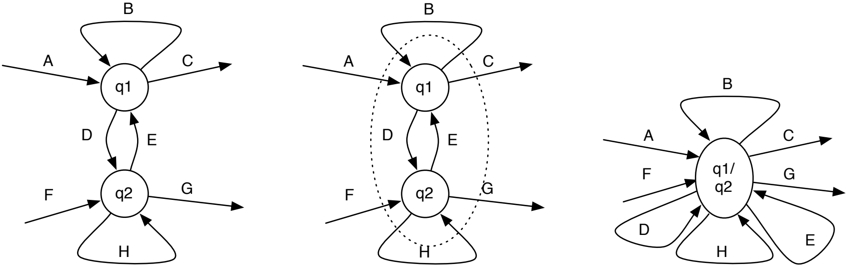
Intersection: Product Automaton



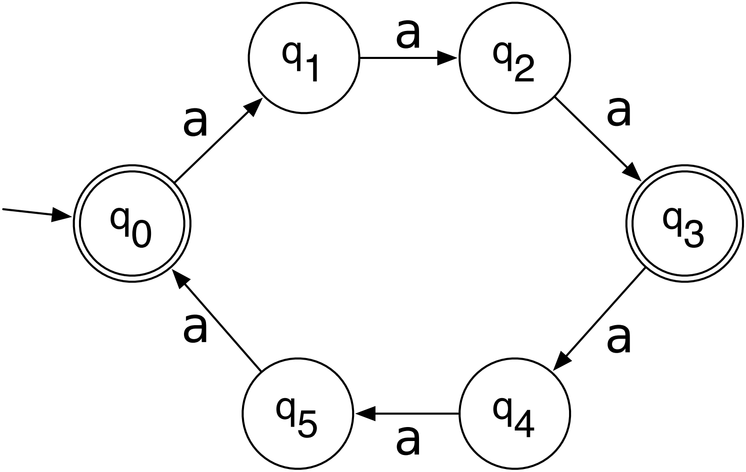
$$\text{DFA } M = (\Sigma, K, \rightarrow, q_0, A)$$
$$\text{DFA } M' = (\Sigma, K', \rightarrow', q_0', A')$$
$$\text{DFA } M \cap M' = (\Sigma, K \times K', \rightarrow_{\text{both}}, \langle q_0, q_0' \rangle, A \times A').$$

State Machine Optimization

If two states have the same language, they can be merged without changing the language of the state machine.



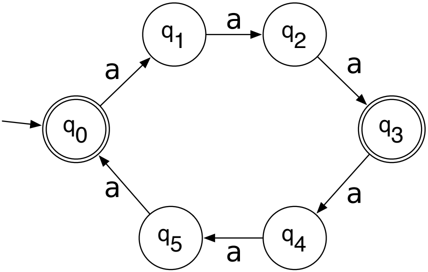
Example



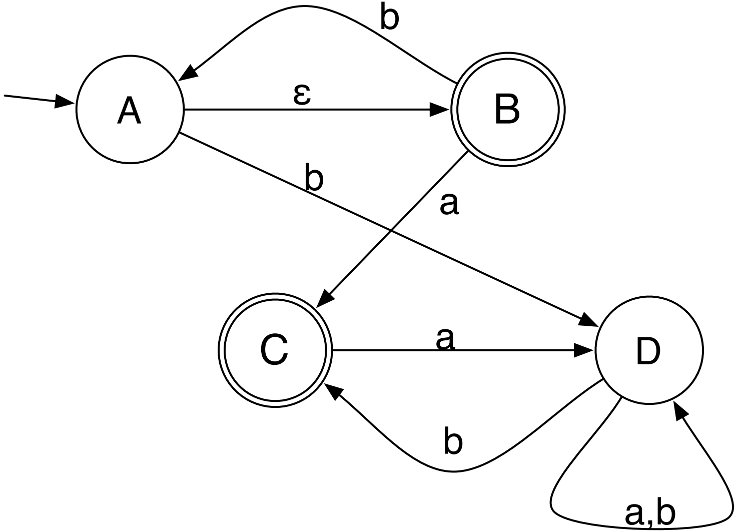
One DFA Minimization Algorithm

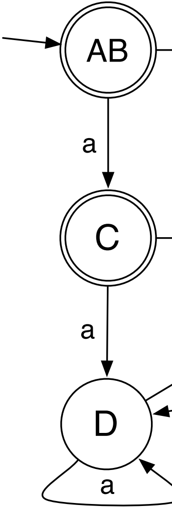
Assume all states are mergable unless there's evidence otherwise:

- ✓ Accepting vs. nonaccepting.
- ✓ Same-symbol transitions to known-different states.

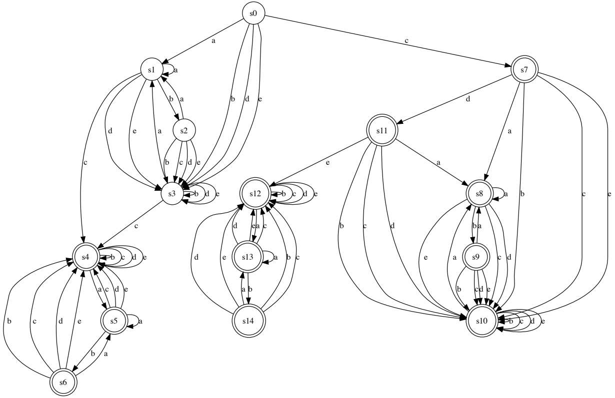


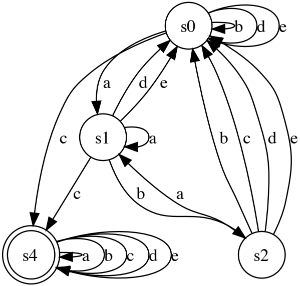
More Complex Example





Bigger Example





Are All Problems Solveable by Computer?

- ✓ Every *real* computer can be accurately modeled as a finite state machine.
- ✓ There are countably many finite state machines.
- ✓ There are uncountably many languages (decision problems)
- ✓ So, no.

But can we find specific examples?

Recall: Derivatives of a Language

For any language L and $x \in \Sigma^*$, define

$$\partial_x L := \{ y \in \Sigma^* \mid xy \in L \}$$

In terms of a state machine for L, the set $\partial_x L$ contains strings that will be accepted after you've already seen x.

So, if you run x through a state machine for L, you end up in a state whose language is $\partial_x L$.

If our state machine is minimal, we should have exactly one state whose language is $\partial_x L$.

Consequence

Theorem (Myhill-Nerode (essentially))

A language is regular iff $\{ \partial_x L \mid x \in \Sigma^* \}$ *is finite.*

Proof idea: The size of this set is the size of the smallest deterministic state machine.

Consider

✓ $L := \{ a^{3n} \mid n \geqslant 0 \}$

$$\partial_\varepsilon L = \partial_{aaa} L = \cdots, \quad \partial_a L = \partial_{aaaa} L = \cdots, \quad \partial_{aa} L = \partial_{aaaaa} L = \cdots$$

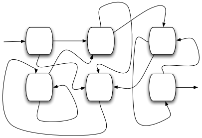
✓ $L := \{ a^n b^n \mid n \geqslant 0 \}$

$$\partial_\varepsilon L \neq \partial_a L \neq \partial_{aa} L \neq \partial_{aaa} L \neq \cdots$$

So, this language is provably not regular!

Maze Theory

- ✓ Suppose you are in a maze of twisty little passages, all alike (but with doors that open from only one side).



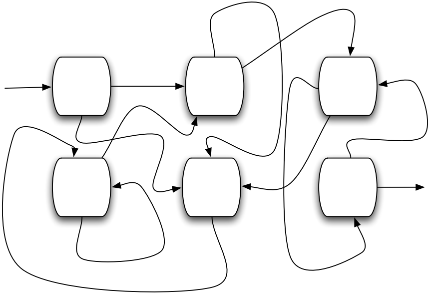
- ✓ You happen to know that your maze has exactly 19 rooms. You enter the maze and wander through 27 rooms. What can you conclude?
- ✓ This wandering has brought you to an exit. What can you conclude about other solutions to the maze?
- ✓ Was there anything special about the numbers 19 and 27?

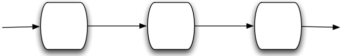
Finite Maze Theorem

For *every* finite maze there is a number p, such that

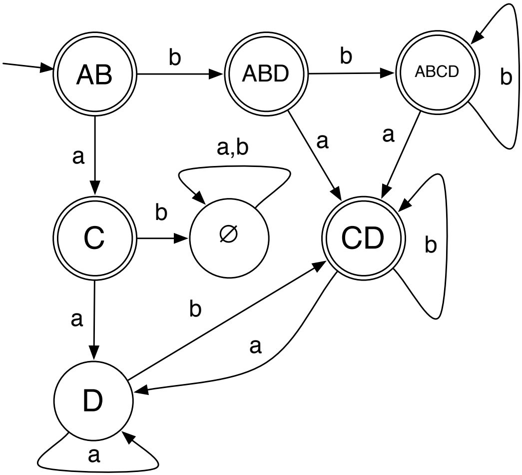
For *every* path through the maze s with $|s| \geqslant p$:

- The path s contains at least one loop, which starts and ends within the first p steps.
- There are infinitely many paths through the maze (at least one shorter, and arbitrarily many longer) whose lengths differ by a multiple of some constant.





Finite Automata As Mazes



A Pumping Lemma

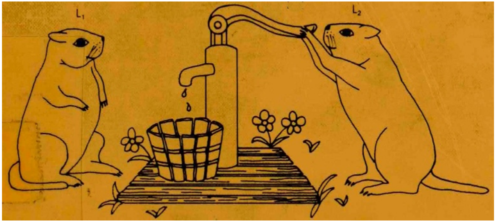
If L is a regular language, then

there exists a number p such that

For every $s \in L$ with $|s| \geq p$

we can decompose s into xyz where

1. $y \neq \varepsilon$
2. $|xy| \leq p$
3. $xy^iz \in L$ for every $i \geq 0$.



Deriving a Useful Corollary

The Pumping Lemma tells us that:

If L is regular,

then every long-enough string in L can be pumped.

What logically follows?

✓ If there's a long string in L that can't be pumped, then L isn't regular!

Using the Pumping Lemma

To prove a language *isn't* regular:

- ✓ Suppose L were regular, with pumping length p
  - Carefully pick a long ($\geqslant p$) string $s \in L$
  - Show that s cannot be pumped
- ✓ Contradiction. Therefore, L is not regular.

You *cannot* use it to prove a language is regular!

- ✓ E.g., non-regular languages with *every* string pumpable

$$\{a^i b^j c^j \mid i \geqslant 1, j \geqslant 0\} \cup \{b^j c^k \mid j, k \geqslant 0\} \qquad p = 1$$

$L = \{ 0^n 1^n \mid n \geqslant 0 \}$

Suppose L were regular

- ✓ Let p be the pumping length
- ✓ Consider, for example, $s := 0^p 1^p$. (Note that $|s| \geqslant p$.)
- ✓ Consider *all* possible decompositions

$$s = xyz \quad \text{with} \quad y \neq \varepsilon \ \wedge \ |xy| \leqslant p.$$

- ✓ None of them work for pumping.
- ✓ Contradiction.

So L is not regular. QED.

Prove nonregular

- ✓ $L = \{ w \in \{0,1\}^* \mid w \text{ has as many 0's as 1's} \}$
- ✓ $L = \{ ww \mid w \in \{0,1\} \}$
- ✓ $L = \{ 0^i 1^j \mid i < j \}$
- ✓ $L = \{ 0^i 1^j \mid i > j \}$