Languages, Reductions, and Undecidability

November 28, 2012
CS 81: Computability and Logic

An engineer and a mathematician are out for a walk and spot a house on fire. There is a garden hose lying in the yard; the engineer hooks it up and puts out the fire. They continue walking, and see a home where a couple is washing their car in their driveway. The mathematician detaches their hose and sets their house on fire, thus reducing it to a previously solved problem.

Church-Turing Thesis

If it can be done at all, then (with suitably coded inputs and outputs) it can be done by

- ✓ A Turing Machine
- ✓ Lambda Calculus
- ✓ An Unrestricted Grammar
- ✓ A 2-register machine
- ✓ Java
- ✓ Scheme
- ✓ Python
- ✓ . . .

Consequently, in most cases where I say "TM," you can think "Program."

TMs and Languages

- ✓ A TM **accepts** a string if it halts saying "yes""
- ✓ A language is **semidecidable** (a.k.a. recognizable, recursively enumerable) if there is a TM that accepts exactly the strings in the language.
- ✓ A language is **decidable** (a.k.a. recursive) if it is accepted by a TM that always halts (i.e., the TM always says "yes" or "no").

Decidable vs. Semidecidable

- ✓ If a language is decidable, then its complement is decidable. Why?

- ✓ If a language is semidecidable, and its complement is semidecidable, then the language is decidable. Why?

Languages of Acceptance

Which are semidecidable (by a TM)? Decidable?

- ✓ $A_{\mathrm{DFA}} = \{ \langle D, w \rangle \mid D \text{ a DFA, } D \text{ accepts } w \}$
- ✓ $A_{\mathrm{NFA}} = \{ \langle N, w \rangle \mid N \text{ an NFA, } N \text{ accepts } w \}$
- ✓ $A_{\mathrm{RE}} = \{ \langle R, w \rangle \mid R \text{ a regexp, } R \text{ matches } w \}$
- ✓ $A_{\mathrm{CFG}} = \{ \langle G, w \rangle \mid G \text{ a CFG, } G \text{ produces } w \}$
- ✓ $A_{\mathrm{TM}} = \{ \langle M, w \rangle \mid M \text{ a TM, } M \text{ accepts } w \}$

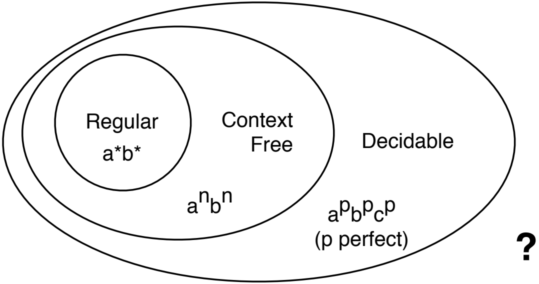
Semidecidability

Show that these languages are semidecidable.

- ✓ Accepts-s:= $\{\ \langle M \rangle \mid M \text{ accepts } s\ \}$
- ✓ $NE_{TM} := \{\ \langle M \rangle \mid M \text{ accepts at least one } w \in \Sigma^*\ \}$

Showing a language **not** semidecidable requires a different approach. (See the Homework.)

Is There More?



Digression: Bootstrapping a Compiler

Lots of compilers are written in the same language they compile!

- ✓ Gnu C Compiler (used in CS 105) is written in C
- ✓ Glasgow Haskell Compiler (used in CS 131) is in Haskell

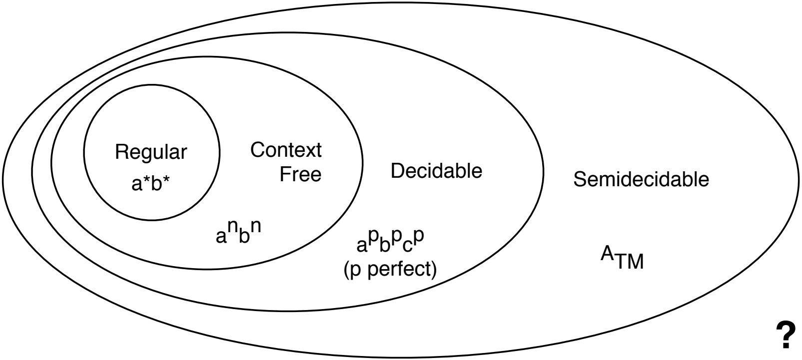
Practical reasons to run programs on their own source code!

A_{TM} IS NOT DECIDABLE

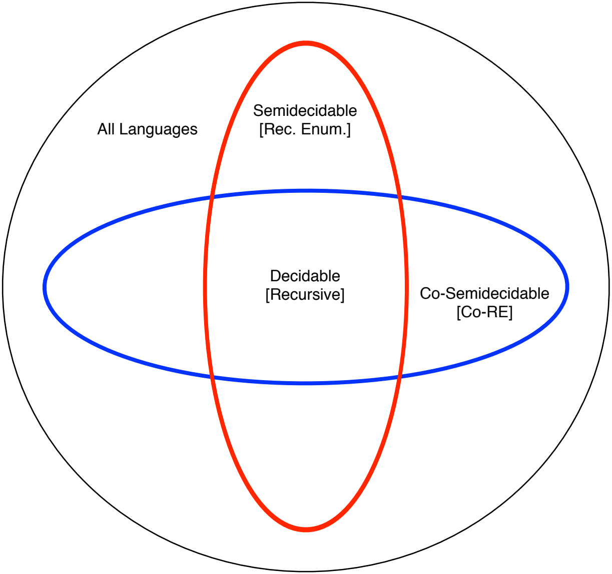
$$A_{TM} = \{\, \langle M, w\rangle \mid M \text{ a TM}, M \text{ accepts } w \,\} A$$

	$\langle M_0\rangle$	$\langle M_1\rangle$	$\langle M_2\rangle$	$\langle M_3\rangle$	$\langle M_4\rangle$	$\cdots$
M_0	Acc.		Acc.	Acc.		
M_1						
M_2	Acc.			Acc.		
M_3	Acc.	Acc.	Acc.	Acc.	Acc.	
M_4			Acc.	Acc.		

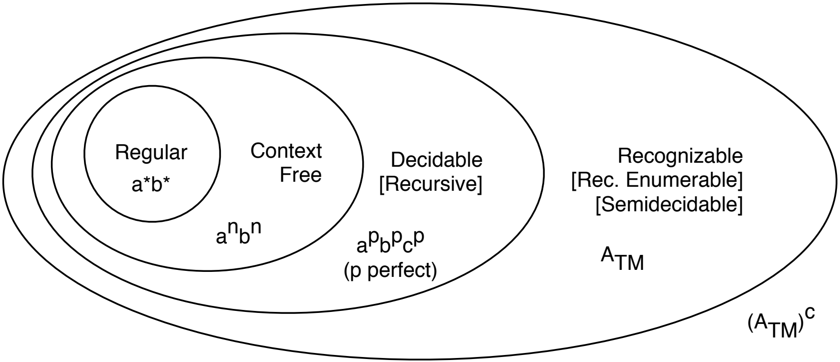
Is There More?



WHAT IS THE COMPLEMENT OF A_{TM}?



We'll Stop Here



Obligatory Corollary

Theorem

The language

$$H = \{ \langle M, w \rangle \mid M \text{ a TM}, M \text{ halts on } w \}$$

is not decidable.

Proof.

Suppose there were a halt-checking TM… □

Undecidability, So Far

1. Acceptance for TMs is semidecidable but not decidable.

$$A_{\mathsf{TM}} = \{\, \langle M, w \rangle \mid M \text{ a TM}, M \text{ accepts } w \,\}$$

2. *Non*-Acceptance for TMs is not semidecidable (hence not decidable).

3. TM Halting is semidecidable but not decidable.

$$H = \{\, \langle M, w \rangle \mid M \text{ a TM}, M \text{ halts on } w \,\}$$

Problem Hardness

If I proved $\mathrm{P} = \mathrm{NP}$ then I would become rich and famous.

Therefore, being rich and famous can't be harder than proving $\mathrm{P} = \mathrm{NP}$!

Rich & Famous $\leqslant$ Prove $\mathrm{P} = \mathrm{NP}$

Reductions

Given problems P and Q, *we say that*

$$P \leqslant Q$$

if a solution to Q would let us solve P as well.

- ✓ I.e., (P is "not fundamentally more difficult" Q.)
- ✓ We say "P **reduces to** Q."

In Math, *we often show we can solve a problem* X by taking advantage of previously-solved problem Q (i.e., prove $X \leqslant Q$)

In Theocomp, *we typically prove a problem* X *hard* by showing a solution to X would also solve the hard problem P (i.e., prove $P \leqslant X$).

Reductions

To prove that P reduces to Q ($P \leqslant Q$), it suffices to prove:

- ✓ If we have a solver for Q, then we can use it to solve any instance of P.
- ✓ I.e., show you could construct a P-solver if you could make calls to a Q-solving subroutine.

Commonly, we instead prove a "mapping reduction":

- ✓ For every instance of P, we can contruct an instance of Q with the same yes/no answer.

Why is this enough?

Warning

It is easy to get the reduction backwards!

Correct form:

- ✓ Assume the unknown problem X is decidable
- ✓ Show that it means that H (or A_{TM} or $\ldots$) is decidable.
- ✓ Contradiction.
- ✓ Therefore, X is not decidable

Or:

- ✓ Assume the unknown problem X is semidecidable
- ✓ Show that it means that $\neg H$ (or $\neg A_{TM}$ or $\ldots$) is semidecidable.
- ✓ Contradiction.
- ✓ Therefore, X is not semidecidable.

If you ever find yourself *assuming* things previously proved impossible ("assume I had a way to decide halting…then X is decidable.") you're doing the reduction wrong!

Reduction Practice

Show the following are not decidable (e.g., by reducing A_{TM} **to** each).

- ✓ $NE_{TM} := \{ \langle M \rangle \mid M \text{ accepts at least one } w \in \Sigma^* \}$
- ✓ $E_{TM} := \{ \langle M \rangle \mid L(M) = \emptyset \}$
- ✓ $ALL_{TM} := \{ \langle M \rangle \mid L(M) = \Sigma^* \}$
- ✓ $\text{Accepts-s} := \{ \langle M \rangle \mid M \text{ accepts } s \}$
- ✓ $\text{Regular} := \{ \langle M \rangle \mid L(M) \text{ is regular} \}$