$(p \to q \to r) \to (p \to q) \to p \to r$

1.	$p \to q \to r$	assumption
2.	$p \to q$	assumption
3.	p	assumption
4.	q	MP 2,3
5.	$q \to r$	MP 3,1
6.	r	MP 5,4
7.	$p \to r$	$\to$i 3-6
8.	$(p \to q) \to p \to r$	$\to$i 2-7
9.	$(p \to q \to r) \to (p \to q) \to p \to r$	$\to$i 1-8

Propositional Logic: Proofs and Truth

CS 81: Computability and Logic
September 12, 2012

The Rules So Far

$$\frac{A \qquad B}{A \wedge B}\ \wedge i \qquad\qquad \frac{A \wedge B}{A}\ \wedge e1 \qquad\qquad \frac{A \wedge B}{B}\ \wedge e2$$

$$\frac{A}{A \vee B}\ \vee i1 \qquad\qquad \frac{B}{A \vee B}\ \vee i2 \qquad\qquad \frac{A \vee B \qquad \boxed{\begin{array}{c} A \\ \vdots \\ C \end{array}} \qquad \boxed{\begin{array}{c} B \\ \vdots \\ C \end{array}}}{C}\ \vee e$$

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{A \to B}\ \to i \qquad\qquad \frac{A \to B \qquad A}{B}\ \to e\ \text{(MP)}$$

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg i \qquad\qquad \frac{\neg A \qquad A}{\bot}\ \neg e \qquad\qquad \frac{}{\top}\ \top i \qquad\qquad \frac{\bot}{A}\ \bot e$$

Prove

$\vdash \neg(p \wedge \neg p)$

$\vdash p \rightarrow \neg\neg p$

$p \rightarrow q, \neg q \vdash \neg p$

$\neg p \vee q \vdash p \rightarrow q$

If $x \in (S \cup T) \cap U$ then $x \in (S \cap U) \cup (S \cap U)$.

Express in terms of $x \in S$ and $x \in T$ and $x \in U$

Natural Deduction Proof?

Assume $x \in (S \cup T) \cap U$, so that $x \in (S \cup T)$ and $x \in U$.
If $x \in S$, then $x \in S \cap U$, and hence $x \in (S \cap U) \cup (T \cap U)$.
The other case is similar. QED.

The Rules So Far

$$\frac{A \qquad B}{A \wedge B}\ \wedge i \qquad\qquad \frac{A \wedge B}{A}\ \wedge e1 \qquad\qquad \frac{A \wedge B}{B}\ \wedge e2$$

$$\frac{A}{A \vee B}\ \vee i1 \qquad\qquad \frac{B}{A \vee B}\ \vee i2 \qquad\qquad \frac{A \vee B \qquad \boxed{\begin{array}{c} A \\ \vdots \\ C \end{array}} \qquad \boxed{\begin{array}{c} B \\ \vdots \\ C \end{array}}}{C}\ \vee e$$

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{A \rightarrow B} \rightarrow i \qquad\qquad \frac{A \rightarrow B \qquad A}{B} \rightarrow e\ (\mathrm{MP})$$

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg i \qquad\qquad \frac{\neg A \qquad A}{\bot}\ \neg e \qquad\qquad \frac{}{\top}\ \top i \qquad\qquad \frac{\bot}{A}\ \bot e$$

Nonconstructive Proof

(repeated in the textbook)

Even the Greeks knew that $\sqrt{2}$ is irrational.

Consider $\sqrt{2}^{\sqrt{2}}$. Either it's rational, or it's not.

- If it is rational, we're done (with $a = b = \sqrt{2}$).

- If it's irrational, we're done (with $a = \sqrt{2}^{\sqrt{2}}$, $b = \sqrt{2}$).

In either case, one can find irrational a and b such that a^b is rational

QED.

Constructive Proof

(not in the textbook)

Put $a := \sqrt{2} = 2^{\frac{1}{2}}$. It's still irrational.

Put $b := \log_2 9 = 2\log_2 3$. It's irrational too.

But $a^b = (2^{\frac{1}{2}})^{2\log_2 3} = 2^{\left(\frac{1}{2}\times 2\log_2 3\right)} = 2^{\log_2 3} = 3$

QED.

Classical Propositional Logic

$$\dfrac{A \qquad B}{A \wedge B}\,\wedge i \qquad\qquad \dfrac{A \wedge B}{A}\,\wedge e1 \qquad\qquad \dfrac{A \wedge B}{B}\,\wedge e2$$

$$\dfrac{A}{A \vee B}\,\vee i1 \qquad\qquad \dfrac{B}{A \vee B}\,\vee i2 \qquad\qquad \dfrac{A \vee B \qquad \boxed{\begin{array}{c}A\\ \vdots\\ C\end{array}} \qquad \boxed{\begin{array}{c}B\\ \vdots\\ C\end{array}}}{C}\,\vee e$$

$$\dfrac{\boxed{\begin{array}{c}A\\ \vdots\\ B\end{array}}}{A \rightarrow B} \rightarrow i \qquad\qquad \dfrac{A \rightarrow B \qquad A}{B} \rightarrow e\ (\mathrm{MP})$$

$$\dfrac{\boxed{\begin{array}{c}A\\ \vdots\\ \bot\end{array}}}{\neg A}\,\neg i \qquad\qquad \dfrac{\neg A \qquad A}{\bot}\,\neg e \qquad\qquad \dfrac{}{\top}\,\top i \qquad\qquad \dfrac{\bot}{A}\,\bot e$$

$$\dfrac{\neg\neg A}{A}\,\neg\neg e \qquad\qquad \dfrac{\boxed{\begin{array}{c}\neg A\\ \vdots\\ \bot\end{array}}}{A}\,\mathrm{PBC} \qquad\qquad \dfrac{}{A \vee \neg A}\,\mathrm{LEM}$$

Prove

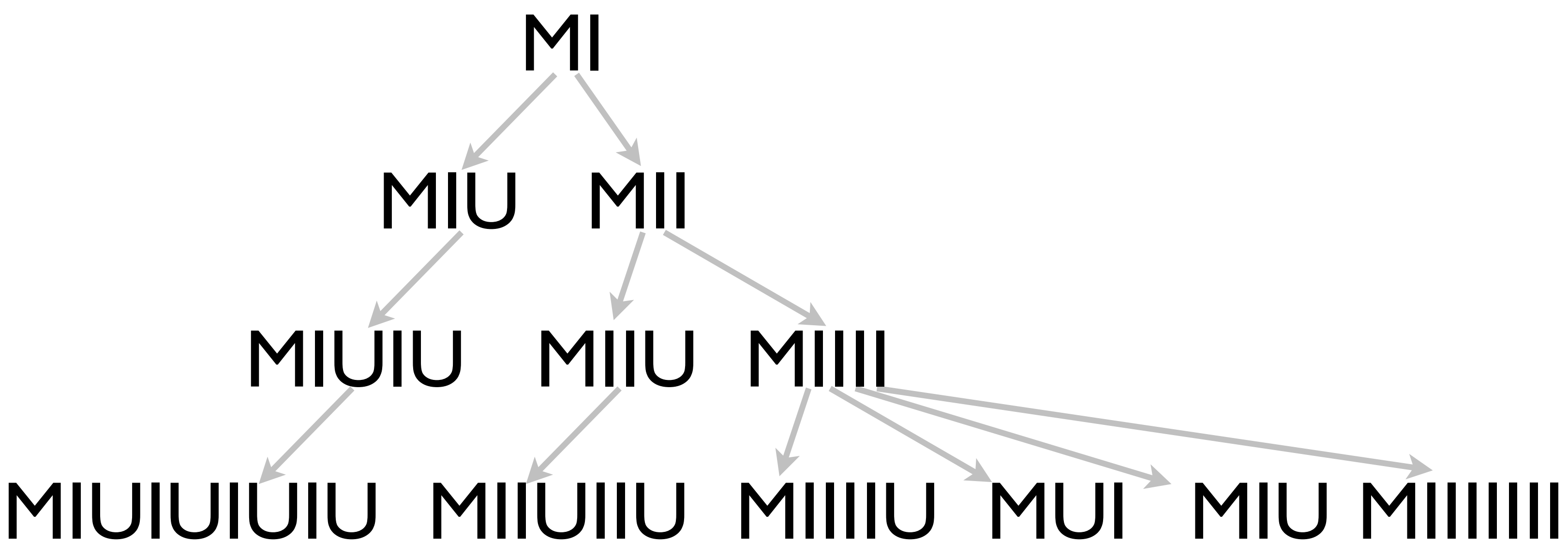
$p \rightarrow q \vdash \neg p \vee q$

$\neg\neg p \rightarrow p$

Semantics of Natural Deduction

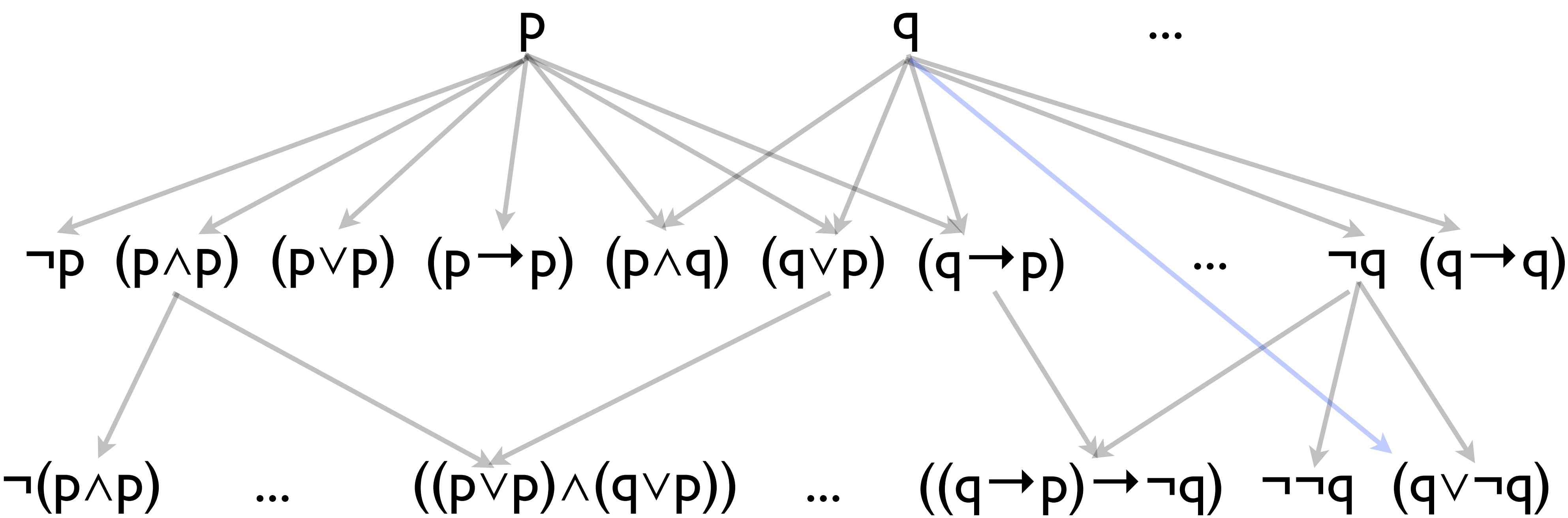
CS 81: Computability and Logic

MIU System



...etc...

Example: Propositional Formulas



...etc...

Natural Deduction Rules

$$\frac{A \qquad B}{A \wedge B}\,\wedge i \qquad\qquad \frac{A \wedge B}{A}\,\wedge e1 \qquad\qquad \frac{A \wedge B}{B}\,\wedge e2$$

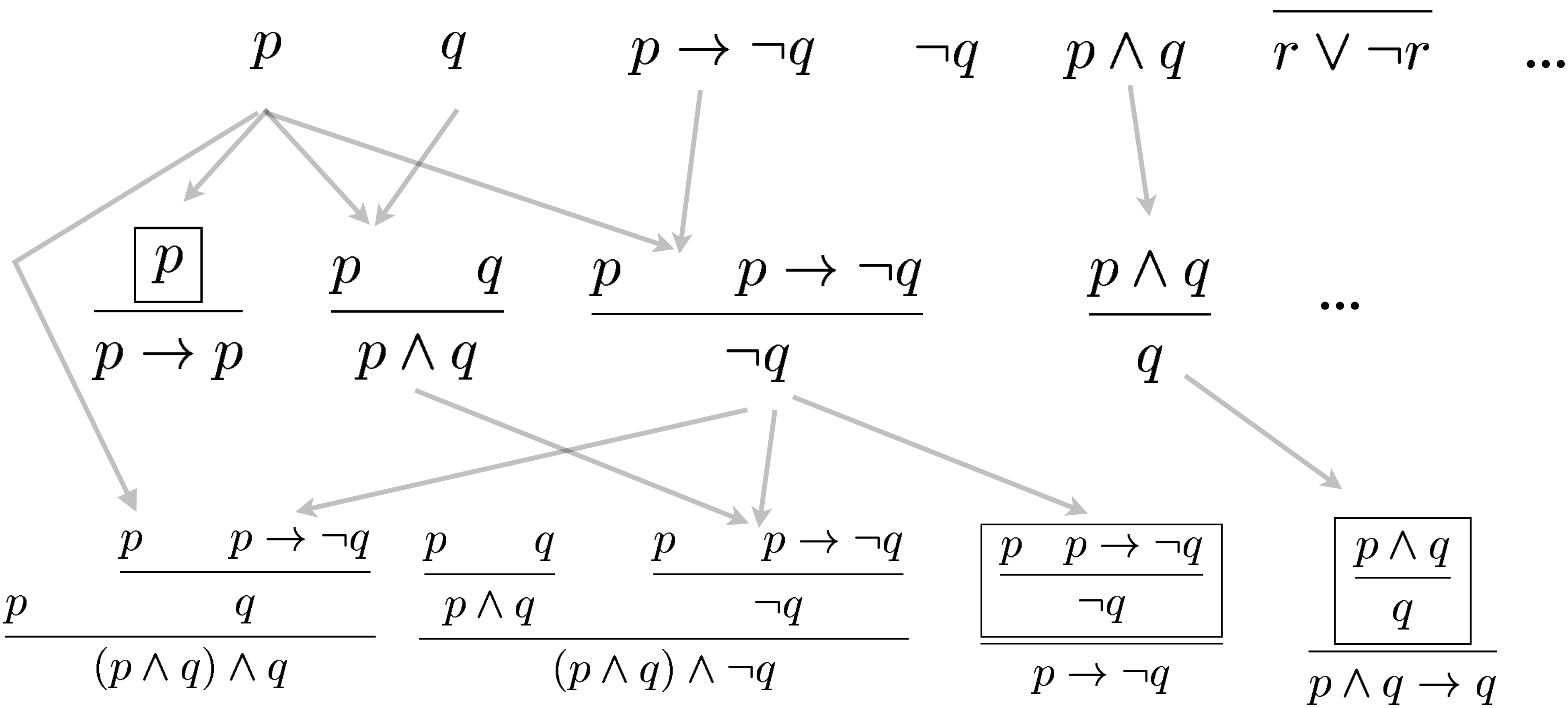
$$\frac{A}{A \vee B}\,\vee i1 \qquad\qquad \frac{B}{A \vee B}\,\vee i2 \qquad\qquad \frac{A \vee B \qquad \boxed{\begin{array}{c} A \\ \vdots \\ C \end{array}} \qquad \boxed{\begin{array}{c} B \\ \vdots \\ C \end{array}}}{C}\,\vee e$$

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{A \to B} \to i \qquad\qquad \frac{A \to B \qquad A}{B} \to e\ (\mathrm{MP})$$

$$\frac{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}{\neg A}\,\neg i \qquad\qquad \frac{\neg A \qquad A}{\bot}\,\neg e \qquad\qquad \frac{}{\top}\,\top i \qquad\qquad \frac{\bot}{A}\,\bot e$$

$$\frac{\neg\neg A}{A}\,\neg\neg e \qquad\qquad \frac{\boxed{\begin{array}{c} \neg A \\ \vdots \\ \bot \end{array}}}{A}\,\mathrm{PBC} \qquad\qquad \frac{}{A \vee \neg A}\,\mathrm{LEM}$$

Proofs *with Assumptions*



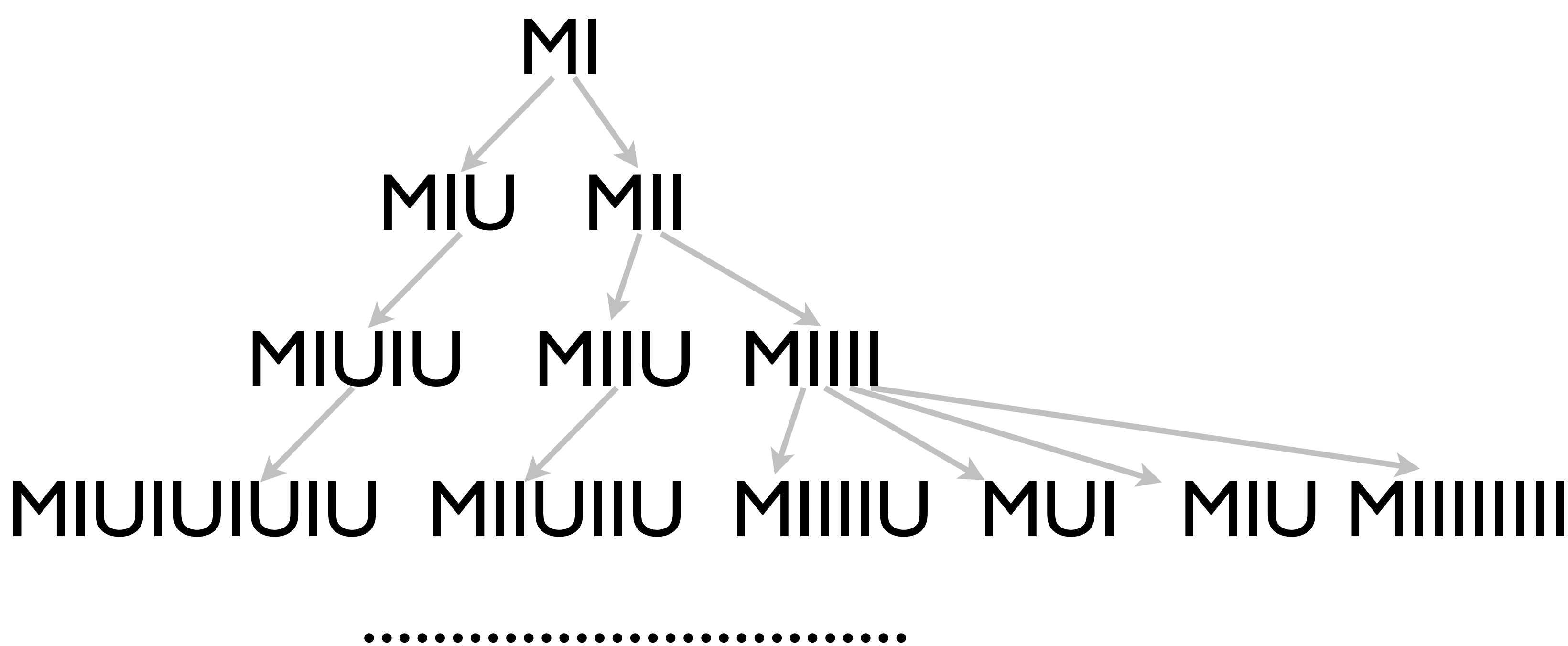
...etc...

Syntax vs. Semantics

We can always do “symbol manipulation”

But is there any <u>meaning</u> to our rules?

MIU System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Hofstadter's pq- System

Axiom Scheme (For any nonempty string of hyphens x):

$\vdash x$p-qx-

Rule of Inference (For hyphen strings x, y and z):

If $\vdash x$pyqz then $\vdash x$py-qz-.

Natural Deduction Rules

$$\frac{A \quad B}{A \wedge B} \wedge_i \qquad \frac{A \wedge B}{A} \wedge_{e1} \qquad \frac{A \wedge B}{B} \wedge_{e2}$$

$$\frac{A}{A \vee B} \vee_{i1} \qquad \frac{B}{A \vee B} \vee_{i2} \qquad \frac{A \vee B \quad \begin{matrix} A \\ \vdots \\ \overline{C} \end{matrix} \quad \begin{matrix} B \\ \vdots \\ \overline{C} \end{matrix}}{C} \vee_e$$

$$\frac{\begin{matrix} A \\ \vdots \\ B \end{matrix}}{A \to B} \to_i \qquad \frac{A \to B \quad A}{B} \to_e \text{(MP)}$$

$$\frac{\begin{matrix} A \\ \vdots \\ \bot \end{matrix}}{\neg A} \neg_i \qquad \frac{\neg A \quad A}{\bot} \neg_e \qquad \frac{}{\top} \top_i \qquad \frac{\bot}{A} \bot_e$$

$$\frac{\neg\neg A}{A} \neg\neg_e \qquad \frac{\begin{matrix} \neg A \\ \vdots \\ \bot \end{matrix}}{A} \text{PBC} \qquad \frac{}{A \vee \neg A} \text{LEM}$$

Meanings of the Connectives

A	¬A
T	F
F	T

⊥
F

⊤
T

A	B	A∧B
F	F	F
F	T	F
T	F	F
T	T	T

A	B	A∨B
F	F	F
F	T	T
T	F	T
T	T	T

A	B	A→B
F	F	T
F	T	T
T	F	F
T	T	T

So: what is the meaning of "(p → (¬p∧q))" ?

Valuations/Models

$$v: \{ p, q, r, \ldots\} \rightarrow \{T,F\}.$$

E.g., $v_1(p) = T$, $v_1(q) = F$, $v_1(r) = T$, ...

or $v_2(p) = F$, $v_2(q) = F$, $v_2(r) = F$, ...

By "abuse of notation" I write

$$v(A)$$

to denote the *meaning* of a formula A.

Satisfiability

A formula is satisfiable if *some* valuation makes it true.

A formula is unsatisfiable if *no* valuation makes it true.

A formula is a tautology if *all* valuations make it true.

Equivalently, if its negation is ...

Semantic Entailment

$$B, C, D \models A$$

Whenever B, C, and D are <u>true</u>, so is A.

==

Every valuation that makes B, C, and D true also makes A true. (Other valuations might not)

Truth vs. Provability

A is true if B & C & D are:

$$B,C,D \models A$$

A is provable if B & C & D are:

$$B,C,D \vdash A$$

Soundness: Provability implies Truth

Completeness: Truth implies Provability

So, if it's not true, don't try to prove it!