Propositional Logic: Proofs vs. Truth

CS 81: Computability and Logic
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The Rules So Far

$$\frac{A \qquad B}{A \wedge B}\ \wedge i \qquad\qquad \frac{A \wedge B}{A}\ \wedge e1 \qquad\qquad \frac{A \wedge B}{B}\ \wedge e2$$

$$\frac{A}{A \vee B}\ \vee i1 \qquad\qquad \frac{B}{A \vee B}\ \vee i2 \qquad\qquad \frac{A \vee B \qquad \boxed{\begin{array}{c} A \\ \vdots \\ C \end{array}} \qquad \boxed{\begin{array}{c} B \\ \vdots \\ C \end{array}}}{C}\ \vee e$$

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ B \end{array}}}{A \to B}\ \to i \qquad\qquad \frac{A \to B \qquad A}{B}\ \to e\ (\text{MP})$$

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg i \qquad\qquad \frac{\neg A \qquad A}{\bot}\ \neg e \qquad\qquad \frac{}{\top}\ \top i \qquad\qquad \frac{\bot}{A}\ \bot e$$

Meanings of the Connectives

A	¬A
T	F
F	T

⊥
F

⊤
T

A	B	A∧B
F	F	F
F	T	F
T	F	F
T	T	T

A	B	A∨B
F	F	F
F	T	T
T	F	T
T	T	T

A	B	A→B
F	F	T
F	T	T
T	F	F
T	T	T

So: what is the meaning of "(p → (¬p∧q))" ?

Satisfiability

A formula is <u>satisfiable</u> if *some* valuation makes it true.

A formula is <u>unsatisfiable</u> if *no* valuation makes it true.

A formula is a <u>tautology</u> if *all* valuations make it true.

Equivalently, if its negation is ...

Semantic Entailment

$$B, C, D \models A$$

Whenever B, C, and D are true, so is A.

==

Every valuation that makes B, C, and D true also makes A true. (Other valuations might not)

Truth vs. Provability

A is true if B & C & D are:

$$B,C,D \models A$$

A is provable if B & C & D are:

$$B,C,D \vdash A$$

Soundness: Provability implies Truth

Completeness: Truth implies Provability

So, if it's not true, don't try to prove it!

Exercise

Which of the following are provable?

$\vdash p \to \neg q \to \neg p$

$\vdash p \to \neg p \to \neg p$

$(p \vee q) \wedge r \vdash (p \wedge r) \vee (q \wedge r)$

$p \to q \vdash p \to \neg q \to \neg p$

$p \to q \vdash (p \to \neg q) \to \neg p$

Consequences

There's an **algorithm** for determining provability (or non-provability) in classical propositional logic.

There's even an algorithm for creating a (horrible) proof when the formula is provable.

Everything you already know about truth tables and boolean logic still applies.

Soundness: If $\vdash$ then $\models$

Strategy: proof by induction <u>on proof trees</u>!

We want to show for every proof tree:

If a valuation makes its *premises* true

Then that valuation makes the *conclusion* true.

$$\dfrac{\dfrac{p \qquad q}{p \wedge q} \qquad \dfrac{p \qquad p \to \neg q}{\neg q}}{(p \wedge q) \wedge \neg q}$$

To show: If a valuation makes the proof's *premises* true, then it also makes the *conclusion* true.

Case

The proof has the form

$$\frac{\begin{matrix}\vdots \\ A\end{matrix} \qquad \begin{matrix}\vdots \\ B\end{matrix}}{A \wedge B} \wedge_i$$

Assume we have a valuation that makes the premises of this *proof-as-a-whole* true.

Case

To show: If a valuation makes the proof's *premises* true, then it also makes the *conclusion* true.

The proof has the form

$$\dfrac{\begin{matrix}\vdots \\ A\end{matrix} \qquad \begin{matrix}\vdots \\ A \to B\end{matrix}}{B} \to i$$

Assume we have a valuation that makes the premises of this *proof-as-a-whole* true.

...

To show: If a valuation makes the proof's *premises* true, then it also makes the *conclusion* true.

Case

The proof has the form

$$\frac{\begin{array}{c} A \\ \vdots \\ B \end{array}}{A \to B}$$

Assume we have a valuation that makes the premises of this *proof-as-a-whole* true.

...

Completeness: If $\vDash$ then $\vdash$

It suffices to prove "If $\vDash$ C then $\vdash$ C."

Assume $A_1, \ldots, A_n \vDash B$.

1. Show $\vDash A_1 \rightarrow \ldots A_n \rightarrow B$. Easy.

2. Show $\vdash A_1 \rightarrow \ldots A_n \rightarrow B$.

3. $A_1, \ldots, A_n \vdash B$. Easy.

Idea 1

Note: we can generalize this idea for <u>any</u> valuation!

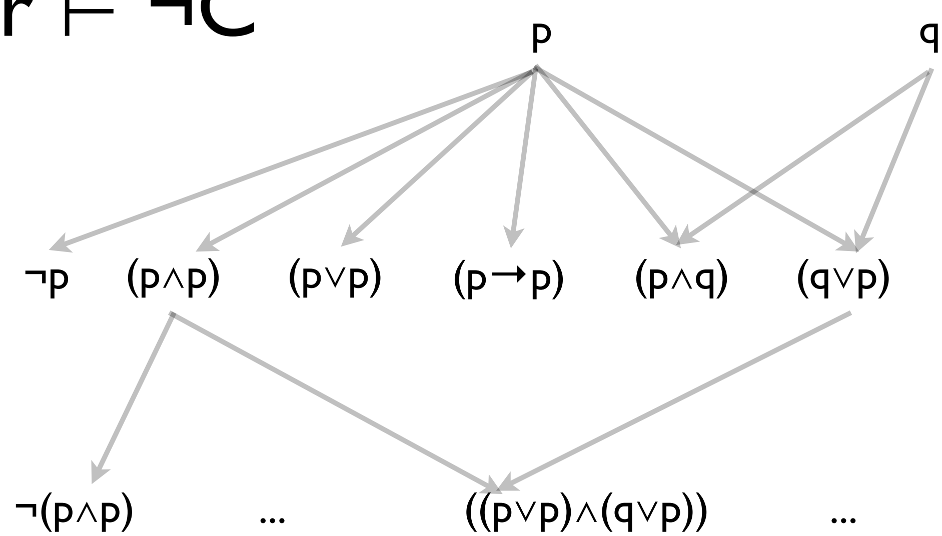
Suppose $v(p) = T \quad v(q) = F \quad v(r) = T$

Claim: For any formula C involving p,q,r:

if $v(C) = T$ then $p, \neg q, r \vdash C$

if $v(C) = F$ then $p, \neg q, r \vdash \neg C$

Proof: By induction on C!



Idea 2

Assume $\vDash$ C.

Use LEM + proof by cases to consider <u>all</u> combinations

p vs. $\neg$p

q vs. $\neg$q

...all other variables in C...

For each of these 2^n cases, use Idea 1 to prove C.

Conclude $\vdash$ C.

Formalizing Arguments

Socrates is a man.

If Socrates is a man, then Socrates is mortal.

$$\frac{\begin{array}{l} p \\ p \rightarrow q \end{array}}{q}$$

p = Socrates is a man

q = Socrates is mortal

Syllogisms: Valid Argument?

All men are mortal

Socrates is a man

Socrates is mortal

p

q

---

r

p = all men are mortal

q = Socrates is a man

r = Socrates is mortal

Your Conclusion?

No interesting poems are unpopular among people of real taste.

No modern poetry is free from affectation.

All your poems are on the subject of soap-bubbles.

No affected poetry is popular among people of real taste.

No ancient poem is on the subject of soap-bubbles.

I = it is interesting
P = it is popular among people of real taste
M = it is modern
A = it is affected
Y = it is your poem
S = it is on the subject of soap bubbles

Problems

I = it is interesting
P = it is popular among people of real taste
M = it is modern
A = it is affected
Y = it is your poem
S = it is on the subject of soap bubbles

Predicate Logic

a.k.a. Predicate Calculus

Individuals

The things we are talking about, e.g.,

- Natural numbers
- Mathematical objects
- People
- Programs
- "Things that could belong to a set"

Terms

(Possibly complicated) references to individuals, e.g.,

- Socrates
- Pittsburgh
- the mother of George Washington
- the author of *Waverly*
- the sum of 2 and 4
- the set of solutions to the equation $x^2+1 = 0$
- $x+y$
- $f(x)$

Term Ingredients

- Variables (x, y, z, ...)
- Constants (a, b, c, ...)
- Function Symbols (f, g, h, ...) with arities
  - successor(n), father-of(Bob), geometric-average(x,y)

Formula Ingredients

- Propositional variables (p, q, ...)
- Predicates/Relations (P, Q, ...) with arities
  - even(n), father(Sue, Bob), n > m
- Familiar logical connectives and constants
- Universal and Existential quantifiers!
  - ∀x. ...
  - ∃x. ...

A(x,y)	x admires y
T(x,y)	x attended y
P(x)	x is a professor
S(x)	x is a student
L(x)	x is a lecture
m	Mary

- Mary admires herself.
- Mary admires every professor.
- Some professor admires Mary.
- Every professor admires Mary.
- No student attended every lecture.
- Mary attended every lecture.
- No lecture was attended by every student.
- No lecture was attended by any student.

Better Translation?

No interesting poems are unpopular among people of real taste.

No modern poetry is free from affectation.

All your poems are on the subject of soap-bubbles.

No affected poetry is popular among people of real taste.

No ancient poem is on the subject of soap-bubbles.

I(x)	x is interesting
P(x)	x is popular among people of real taste
M(x)	x is modern
A(x)	x is affected
Y(x)	x is your poem
S(x)	x is on the subject of soap bubbles