Hoare Logic

CS 81: Computability and Logic
October 1, 2012

State Transformers

State before → `x = x + 1` → State after

precondition	postcondition
`x==1`	`x==2`
`x==7`	`x==8`
`even(x)`	`odd(x)`
`x > 3`	`x > 4`

precondition *postcondition*

Preconditions and Postconditions

precondition → `x = x + 1` → *postcondition*

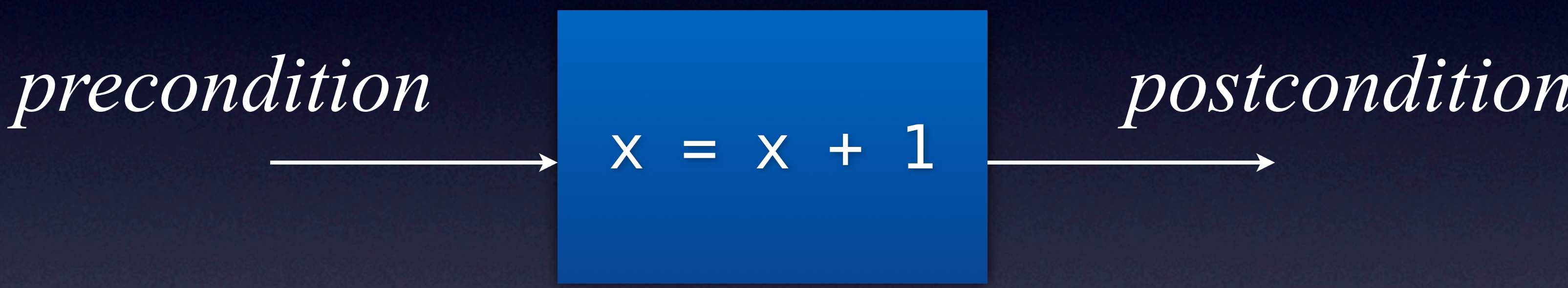
`x == 1`

`x > 10`

$\top$

$\bot$

Preconditions and Postconditions




```
x == 1
x > 10
```

$\top$

$\bot$

Design by Contract

precondition → *implementation* ← *postcondition*

Specification for Sorting?

precondition

int a[N];

postcondition

$$\forall i.\ \ (1 \le i < \mathrm{N}) \rightarrow \mathtt{a}[i\text{-}1] \le \mathtt{a}[i]$$

implementation

```
for (int i=0; i<n; ++i)
    a[i] = 0;
```

Hoare Triples

{ *precondition* } `code` { *postcondition* }

Weakest Preconditions?

{ ??? } $x = x + 2$ { $x > 7$ }

{ ??? } $x = 2$ { $xy > 7$ }

{ ??? } $x = y + z$ { $xy > 7$ }

{ ??? } $x = e$ { $Q(x)$ }

Assignments

$$\frac{}{\{\ P[e/x]\ \}\ \ x\ \texttt{=}\ e\texttt{;}\ \ \{\ P\ \}}\ \textsc{Assignment}$$

But...

Are these provable? Are they “true”?

$$\{\ 49 > 42\ \}\ \ x = 7\ \ \{\ x^2 > 42\ \}$$

$$\{\ \top\ \}\ \ x = 7\ \ \{\ x^2 > 42\ \}$$

$$\{\ 9 < y\ \}\ \ x = 7\ \ \{\ y > x + 2\ \}$$

$$\{\ y > 9\ \}\ \ x = 7\ \ \{\ y > x\ \}$$

$$\frac{}{\{\ P[e/x]\ \}\ \ x\ =\ e;\ \ \{\ P\ \}}\ \text{ASSIGNMENT}$$

Applying “Normal” Logic

$$\frac{P \to P' \qquad \{\ P'\ \}\ c\ \{\ Q'\ \} \qquad Q' \to Q}{\{\ P\ \}\ c\ \{\ Q\ \}}\ \text{IMPLIED}$$

Weakest Precondition?

{ ??? } `x = 3; y = 4` { yz > 7x }

$$\frac{}{\{\ P[e/x]\ \}\ \ x\ \texttt{=}\ e;\ \ \{\ P\ \}}\ \textsc{Assignment}$$

Sequential Code

$$\frac{\{\ P\ \}\ c_1\ \{\ R\ \} \qquad \{\ R\ \}\ c_2\ \{\ Q\ \}}{\{\ P\ \}\ c_1; c_2\ \{\ Q\ \}}\ \textsc{composition}$$

What if Our Conditions Don't Match?

$$\{\ \top\ \}\ \mathsf{c}_1\ \{\ x > 7\ \}$$

$$\{\ x > 5\ \}\ \mathsf{c}_2\ \{\ y = 2\ \}$$

$$\{\ ???\ \}\ \mathsf{c}_1 ; \mathsf{c}_2\ \{\ ???\ \}$$

Huth & Ryan Proof Format (Sequential Code)

$\{ P_1 \}$
c_1
$\{ P_2 \}$
c_2
$\{ P_3 \}$
...
$\{ P_n \}$
c_n
$\{ P_{n+1} \}$

Hint
Work Bottom Up!

Huth & Ryan Proof Format (Sequential Code + Implies)

$\{ P_{1a} \}$
$\{ P_{1b} \}$
c_1
$\{ P_{2a} \}$
c_2
$\{ P_3 \}$
...
$\{ P_n \}$
c_n
$\{ P_{n+1} \}$

Hint
Work Bottom Up!

Example

{ y = 5 } `y = y + 1` { y = 6 }

{ y = 5 }
{ y + 1 = 6 }   implied
`y = y + 1`
{ y = 6 }   assignment

Sequencing Proof

$\{ 4z > 21 \}$ $x = 3;\ y = 4$ $\{ yz > 7x \}$

$\{ 4z > 21 \}$

$\{ 4z > 7 \times 3 \}$

$x = 3$

$\{ 4z > 7x \}$

$y = 4$

$\{ yz > 7x \}$

Exercise: Swap

$\{ x = x_0 \wedge y = y_0 \}$ `t = x; x = y; y = t` $\{ x = y_0 \wedge y = x_0 \}$

If Statements

$$\frac{\{\ P \land e\ \}\ c_1\ \{\ Q\ \} \qquad \{\ P \land \neg e\ \}\ c_2\ \{\ Q\ \}}{\{\ P\ \}\ \texttt{if}\ (e)\texttt{:}\ c_1\ \ \texttt{else:}\ c_2\ \{\ Q\ \}}\ \text{IF}$$

To get from here to here

```
{ P }
if (e):
    { P ∧ e}
    c1
    { Q }
else:
    { P ∧ ¬e}
    c2
    { Q }
{ Q }
```

Need to complete two sub-proofs

Exercise: Max

$\{ \top \}$

```
if (x > y):
   m = x
else:
   m = y
```

$\{ m = \max(x,y) \}$

Invariants

- Conditions that are “preserved” by code
  - Both a precondition and a postcondition

- Loop invariant:
  True before and after every iteration of a loop.

- Representation Invariant
  True before and after every (public) function on some data structure.

While Statement

$$\frac{\{\ I \wedge e\ \}\ c\ \{\ I\ \}}{\{\ I\ \}\ \texttt{while}\ (e)\texttt{:}\ c\ \{\ I \wedge \neg e\ \}}\ \text{WHILE}$$

To get from here to here

```
{ I }
while (e):
    {I ∧ e}
    c
    {I}
{I ∧ ¬e}
```

Need to complete a sub-proof

Invariant?

$\{\, x = 0 \wedge y = 1 \wedge z = 1 \wedge n \geq 1 \}$

```
while ( z < n ):
  y = x + y
  x = y - x
  z = z + 1
```

$\{\, y = \mathrm{fib}(n) \,\}$

Invariant?

$\{ m = m_0 > 0 \wedge n = n_0 > 0 \}$

```
while ( m != n ):
  if ( m < n ):
    n = n-m
  else:
    m = m-n
```

$\{ m = \gcd(m_0, n_0) \}$

Exercise: While

$\{ x \leq n \}$ `while (x < n) x = x+1` $\{ x = n \}$

Note: *Partial* Correctness

$\{ \top \}$

```
while (true):
   x = x+1
```

$\{ y = 99 \}$

$\{ y = 4 \}$

```
while (y != 42):
   b = not b
```

$\{ y = 42 \}$

Total Correctness

=

Partial Correctness

+

Termination

Proving Termination

One approach: Define a *variant*

(non-negative but decreases on each iteration)

Termination?

```
n = 0
while ( n < 99 ):
     x = x+1
     n = n+1
```

Termination?

$\{$ n $= n_0 > 0\}$

```
x = 0;
while ( n > 0 ):
    x = x+1;
    n = n-1;
```

$\{$ x $= n_0$ $\}$

Partial Correctness?

$\{\ n = n_0 > 0 \}$

```
x = 0;
while ( n > 0 ):
    x = x+1;
    n = n-1;
```

$\{\ x = n_0\ \}$

$\{\ n = n_0 > 0 \}$

```
x = 0;
```

$\{\ I\ \}$

```
while ( n > 0 ):
```

$\{\ I\ \}$

```
    x = x+1;
```

$\{\ \dots\ \}$

```
    n = n-1;
```

$\{\ I\ \}$

$\{\ I \wedge n \leq 0\ \}$

$\{\ x = n_0\ \}$

Total Correctness?

$\{ \texttt{m} = m_0 > 0 \wedge \texttt{n} = n_0 > 0 \}$

```
while ( m != n ):
  if ( m < n ):
    n = n-m
  else:
    m = m-n
```

$\{ \texttt{m} = \gcd(m_0, n_0) \}$

$$\frac{\{ P \wedge e \}\ c_1\ \{ Q \} \qquad \{ P \wedge \neg e \}\ c_2\ \{ Q \}}{\{ P \}\ \texttt{if}\ (e)\ c_1\ \texttt{else}\ c_2\ \{ Q \}}\ \text{IF}$$

$\{ \texttt{m} = m_0 > 0 \wedge \texttt{n} = n_0 > 0 \}$

$\{ I \}$

while (m != n):

$\{ I \}$

if (m < n):

$\{ I \wedge m < n \}$

n = n-m

$\{ I \}$

else:

$\{ I \wedge m \geq n \}$

m = m-n

$\{ I \}$

$\{ I \}$

$\{ I \wedge m = n \}$

$\{ \texttt{m} = \gcd(m_0, n_0) \}$

Total Correctness?

$\{ x = 0 \wedge y = 1 \wedge z = 1 \wedge n \geq 1 \}$

```
while ( z < n ):
  y = x + y
  x = y - x
  z = z + 1
```

$\{ y = fib(n) \}$