Worksheet: Models of Predicate Calculus

1. Assume we have a set of function symbols (e.g., f or cos or $+$), constants (e.g., c or 7 or Bob), and predicate/relation symbols (e.g., P or R or <). A *model* is a specification $\mathcal{M}$ consisting of:

   - A *non-empty* mathematical set $\mathcal{A}$ (the "individuals");
   - For each constant symbol c, an element $c^{\mathcal{M}} \in \mathcal{A}$;
   - For each function symbol f with arity n, a mathematical function $f^{\mathcal{M}} : \mathcal{A}^n \to \mathcal{A}$;
   - For each relation symbol R with arity n, a mathematical relation $R^{\mathcal{M}}$ on the set $\mathcal{A}^n$ (or, equivalently, a mathematical function $\mathcal{A}^n \to \{t, f\}$).

   Suppose we have constants c and d, a unary function k and a binary relation R. Give three different models for these ingredients.

2. In each of your three models, which of these statements are true? (Use your intuition; for exists and forall, think about all the elements of your set $\mathcal{A}$)

   - $R(c, k(d))$
   - $R(c, d) \vee R(d, c)$
   - $\exists x.\ R(x, x)$
   - $\forall x.\ R(x, x)$
   - $\forall x.\ R(x, x) \vee \neg R(x, x)$

3. Consider the following model $\mathcal{M}$: the set A consists of the people in this classroom, there is one unary predicate W that is true if the person is awake, and there is one constant prof representing the professor.

    - If we want to check whether $A \wedge B$ is true in a model, what do we need to check?

    - If we want to check whether $W(x)$ is true, what else do we need to know?

    - If we want to know whether $\forall x.\, W(x)$ is true, what do we need to check?

    - If we want to know whether $\exists x.\, W(x)$ is true in a model, what do we need to check?

4. We say that $A_1, A_2, \ldots \models B$ if every model where all the A_i's true also makes B true. As in predicate logic (although the proof is much harder) we have that

$$A_1, A_2, \ldots \models B \qquad \text{if and only if} \qquad A_1, A_2, \ldots \vdash B.$$

Show that the following statements are *not* provable by finding an appropriate model.

- Assumption: $\forall x. T(x, x)$. Conclusion: $\forall x. \forall y. T(x, k)$

- Assumptions: $D(c)$, and $\forall y. C(y) \to D(y)$. Conclusion: $C(c)$.

- Assumption: $\forall x. \forall y. \forall z. \quad M(x, y) \land M(y, z) \to M(x, z)$.
  Conclusion: $\forall x. \forall y. \quad M(x, y) \lor M(y, x)$.