Worksheet: Proofs in Predicate Calculus

Syntax

$$A, B, C \quad ::= \top \mid \bot \mid p \mid \neg A \mid A \wedge B \mid A \vee B \mid A \to B \mid P(t_1, \ldots, t_n) \mid \forall x.\, A \mid \exists x.\, A$$

$$t, u \quad ::= x \mid c \mid f(t_1, \ldots, t_n)$$

Additional Rules: Simplified Form

$$\frac{\begin{array}{|c|}\hline x_0 \\ \vdots \\ P(x_0) \\ \hline\end{array}}{\forall x.\, P(x)}\ \forall i \qquad (x_0 \text{ "fresh"})$$

$$\frac{\forall x.\, P(x)}{P(t)}\ \forall e$$

$$\frac{P(t)}{\exists x.\, P(x)}\ \exists i$$

$$\frac{\exists x.\, P(x) \qquad \begin{array}{|cc|}\hline x_0 & P(x_0) \\ & \vdots \\ & C \\ \hline\end{array}}{C}\ \exists e \qquad (x_0 \text{ "fresh" and not in } C)$$

Additional Rules: General Form

$$\frac{\begin{array}{|c|}\hline x_0 \\ \vdots \\ A[x_0/x] \\ \hline\end{array}}{\forall x.\, A}\ \forall i \qquad (x_0 \text{ "fresh"})$$

$$\frac{\forall x.\, A}{A[t/x]}\ \forall e$$

$$\frac{A[t/x]}{\exists x.\, A}\ \exists i$$

$$\frac{\exists x.\, A \qquad \begin{array}{|cc|}\hline x_0 & A[x_0/x] \\ & \vdots \\ & C \\ \hline\end{array}}{C}\ \exists e \qquad (x_0 \text{ "fresh" and not in } C)$$

1. $A[t/x]$ is the book's notation for substitution, and refers to *the result of* replacing free occurrences of the variable x in A by the term t. What is:

   (a) $\big(\text{Even}(y) \to \text{Even}(y * 2)\big)\,[1{+}1\,/\,y]$

   (b) $\big(z > 3 \vee (\forall z.\, z^2 > z + y + w)\big)\,[7\,/\,z]$

   (c) $\big(z > 3 \vee (\forall z.\, z^2 > z + y + w)\big)\,[2{*}z\,/\,y]$

2. One "specific instance" (of many) of the $\wedge$i rule is:

$$\frac{\forall z.g(z) < 7 \qquad f(y) > 5}{(\forall z.g(z) < 7) \ \wedge \ f(y) > 5}.$$

   Show specific instances of $\forall$i, $\forall e$, $\exists$i, and $\exists e$.

3. **Prove** $\forall x.\, P(x) \vdash \neg\exists x.\, \neg P(x)$.

4. **Prove** $\exists z.\, \forall w.\, P(z, w) \vdash \forall y.\, \exists x.\, P(x, y)$

5. **Prove** $\forall x.\, P(x) \quad \vdash \quad \exists x.\, P(x)$

6. **Prove** $\neg\exists x.\, (B(x) \wedge G(x)), \quad \forall x.\, (D(x) \rightarrow B(x)) \quad \vdash \quad \neg\exists x.\, (D(x) \wedge G(x))$

7. Look again at Prof. Karp's notes on continuity. Recall that $\lim_{x \to x_0} f(x) = L$ would be formally expressed as:

$$\forall \epsilon.\, \epsilon > 0 \to \exists \delta.\, (\delta > 0) \wedge \big(\forall x.\, (0 < |x - x_0|) \wedge (|x - x_0| < \delta) \to |f(x) - L| < \epsilon\big)$$

Consider the *proof* just above Remark 7. What rules of natural deduction correspond to (explicit or implicit) steps of this proof?

8. What is wrong with this proof of $\exists x.P(x),\ \forall x.\, (P(x) \to Q(x)) \vdash \forall x.Q(x)$?

	1	$\exists x.\, P(x)$	Premise
	2	$\forall x.\, (P(x) \to Q(x))$	Premise
x_0	3		
x_0	4	$P(x_0)$	Assumption
	5	$P(x_0) \to Q(x_0)$	$\forall e$ 2
	6	$Q(x_0)$	MP 5,4
	7	$Q(x_0)$	$\exists e$ 1, 4–6
	8	$\forall y.\, Q(y)$	$\forall i$ 3–7