An Example of a Short NP-Completeness Proof

Ran Libeskind-Hadas, April 2012

Minimum Feedback Vertex Set Optimization Problem: Given a directed graph G, what is the least number of vertices that must be removed (along with edges incident upon those vertices) so that the remaining graph contains no directed cycles?

The corresponding decision problem is:

Minimum Feedback Vertex Set (MFVS): Given a directed graph G and a non-negative integer k, is it possible to remove k or fewer vertices (along with edges incident upon those vertices) such that the resulting graph contains no directed cycle?

We claim that MFVS is NP-complete. MFVS is in the class NP because a certificate would be the set of k vertices to remove. We'd remove these vertices from the graph G to build a new graph G'. We'd then run depth-first search from each vertex to test for cycles. This clearly takes polynomial time.

Next, we show MFVS is NP-hard by a reduction from Vertex Cover (VC). An instance of VC is an undirected graph G and a positive integer k. In our reduction, we build a new graph G' with the same vertices as G but with each edge $\{u, v\}$ of G replaced by a pair of oppositely directed edges (u, v) and (v, u). Call this new directed graph G'. Our instance of MFVS is G', k (we use the same k as in the instance of VC).

Clearly this reduction takes polynomial time since it simply copies G but replaces each edge with two edges.

Now, we show that there is a vertex cover of size k in G if and only if there is a subset of k vertices in G' whose removal breaks all cycles. First, assume that there is a vertex cover of size k in G. Now, remove these k vertices from G' along with the edges incident upon them. Notice that for any directed edge $(u, v) \in G'$, at least one of u or v must have been removed, because one of u or v must have been in our vertex cover. Thus, after removing these k vertices and their incident edges from G', no two vertices have an edge between them, and consequently there can be no cycles.

Conversely, assume that there exists a set S of k vertices in G' whose removal breaks all cycles. By construction every pair of vertices u, v that had an edge between them in G have a cycle $(u, v), (v, u)$ in G'. Since the removal of set S broke all cycles in G', for each edge $\{u, v\} \in G$, the set S must contain either u or v (or both). Thus, S is a vertex cover of G.