Advanced Topics in Algorithms
Computer Science 181b
Spring 2012
Homework 3, Due Wednesday, February 8

1. **[15 Points] Rank and Span Envy!**

   (a) Let $M = (E, \mathcal{I})$ be a matroid. Analogous to the definition in a vector space, the *rank* of a set $X \subseteq E$, denoted $r(X)$, is the size of a largest independent subset of X. Let B be a basis of M. Show that $r(B) = r(E) = |B|$. (Remember, a set is a subset of itself.)

   (b) Recall that if V is a vector space and $X \subseteq V$, the *span* of X, denoted $\text{span}(X)$, is defined to be the subspace of all vectors that can be expressed as a linear combination of vectors in X. It is easy to verify that in a vector space, $\text{span}(X)$ is the unique maximal superset of X with the same rank as X.

   If $M = (E, \mathcal{I})$ is a matroid and $X \subseteq E$, the span of X, denoted $\text{span}(X)$, is defined to be any maximal superset Y of X such that $r(Y) = r(X)$. (Remember, a set is a superset of itself.)

      i. In a vector space, each subset of vectors has a unique span. Is this true for matroids? If so, prove it. If not, give a small counterexample.

      ii. Another property of vector spaces is that for any basis B in a vector space E, $\text{span}(B)$ is the entire vector space. Is the corresponding property, that $\text{span}(B) = E$ for any basis B, true in all matroids? If so, prove it. If not, demonstrate a small counterexample.

2. **[35 Points] The Millisoft Matching Problem!** In this problem we investigate a graph matching problem and its corresponding matroid.

   Gill Bates, CEO of Millisoft, has come up with a revolutionary new scheme for matching up pen pals. He plans to offer the service through his MilliSoft Network web site in the near future.

   Assume that there are n subscribers to the service. Ideally, every subscriber in the network will be paired up with exactly one other subscriber. For compatibility reasons, only certain pairs of subscribers can be matched and it therefore may not be possible to find a match for every subscriber. Each subscriber stipulates a single positive fee that they are willing to pay to get matched. Millisoft would like to find a matching that maximizes the total fees that it can collect from the subscribers.

This optimization problem can be represented with a general graph in which vertices correspond to subscribers, edges correspond to subscribers that can potentially be matched, and the weight on each vertex represents the fee that will be paid if that vertex is matched. Our objective is to find a matching that maximizes the sum of the weights on the matched vertices.

(a) Explain why this problem is more general than the matching problem in unweighted general graphs but is less general than the matching problem in weighted general graphs. (That is, show that it contains the former problem as a special case and is also a special case of the latter problem.)

(b) Show that this problem can be solved using the Matroid Greedy Algorithm. Let $M = (E, I)$ be defined as follows: E is the set of all vertices of the graph and $X \subseteq E$ is in I if there exists some matching in which all of the vertices of X are matched (possibly to vertices of $E - X$). Show that this is a matroid and that our problem is that of finding a basis of maximum weight in this matroid.

(c) Since the problem can be formulated as finding a basis of maximum weight in this matroid, we can apply the Matroid Greedy Algorithm (MGA) to solve the problem. The MGA requires that we test sets for independence. Explain how the independence test would work in this context. Rather than devising a special algorithm for the independence test (which would potentially require a challenging proof of correctness), show how the independence test can be performed by constructing an appropriate graph and applying a known correct algorithm on this graph. You'll still need to prove that your method is correct, but the argument will be much nicer because you'll be using the correctness of an "off-the-shelf" algorithm.

(d) What is the running time of your algorithm? Explain. (It needn't be the fastest possible, just polynomial time.)

3. [**15 Points**] **Chromatic Spanning Trees.** The Republic of Shmorbodia is planning to build a new spanning network to span all of its government buildings. Shmorbodia has k different private internet provides, each of which owns some set of the cables. Thus, you can imagine that each edge has a weight (cost) and a color, where the color indicates the provider that owns that cable. (The graph is simple; it is not a multigraph.) In the spirit of fairness, the Republic has set a limit m_i on the number of cables that can be rented from each provider i, $1 \leq i \leq k$. The objective is to find the spanning tree of minimum weight subject to the constraint on the number of cables rented from

each provider. Explain how this problem can be solved using an algorithm mentioned in class or solved in another homework problem.

4. **[35 Points] The Matroid Intersection Theorem.** The Matroid Intersection Theorem states that for any two matroids over the same ground set E, $M_1 = (E, I_1)$ and $M_2 = (E, I_2)$ with rank functions r_1 and r_2, respectively:

$$max_{X \in I_1 \cap I_2} |X| = min_{U \subseteq E} (r_1(U) + r_2(E - U))$$

   (a) Use this theorem to prove König's Matching Theorem that states that in a bipartite graph, the size of a maximum matching is equal to the size of a minimum vertex cover. (That's our third *different* proof of this result!)

   (b) Use the Matroid Intersection Theorem to prove " König's Other Theorem" (aka the König-Rado Theorem) that states that in a bipartite graph, the size of a minimum edge cover is equal to the size of a maximum independent set. Recall that an edge cover is a set of edges that cover all the vertices in the graph. (*Hint:* Use the same matroids as for part (a). It may useful to first show that the complement of a vertex cover is an independent set – a set of vertices no two of which are connected to one another – and thus minimizing the size of a vertex cover is equivalent to maximizing the size of an independent set.)