Computers: What can't they do!
Computers: What can't they do?

Part 3: State Machines Concluded

April 16–17, 2012

CS 60: Principles of Computer Science

Upcoming Schedule

Assignment 10 (Seam Carving) due Monday, April 16

Assignment 11 (Floyd-Warshall, Computability) due Monday, April 23

Mini-Assignment 12 (Uncomputability) due Friday, April 27

Remaining CS courses: CS 5 and 60 again, more slowly!

Math 55 (Discrete Math) ✔		
CS 70 ✔	Data Structures & Program Devel.	Stacks, BSTs, Maps, OOP
CS 81	Computability and Logic	DFAs, Grammars, TMs, Logic
CS 105	Computer Systems	Binary, HMMM
CS 121	Software Development	Inheritance, Helper functions
CS 131	Programming Languages	Racket, Python, Java
CS 140	Algorithms	DFS, Sorting, Dyn. Prog., Big-O
+Three Electives		
CS 132	Compilers	Parsing, HMMM
CS 151	Artificial Intelligence	Prolog, Connect-4, Picobot
CS 152 ✔	Neural Networks	Connect-4
CS 155	Graphics	Seam Carving
CS 181A	Complexity Theory	Big-O
CS 181C ✔	Domain-Specific Languages	Unicalc
CS 181E ✔	Foundations of Parallel Prog.	Threads, Java
⋮	⋮	⋮
+Clinic		

Where We're Going

There are problems that no computer can solve.

What is a **problem**?

- ✓ Input: a finite sequence of symbols (a "string")
- ✓ Output: yes or no.

What is a **computer**?

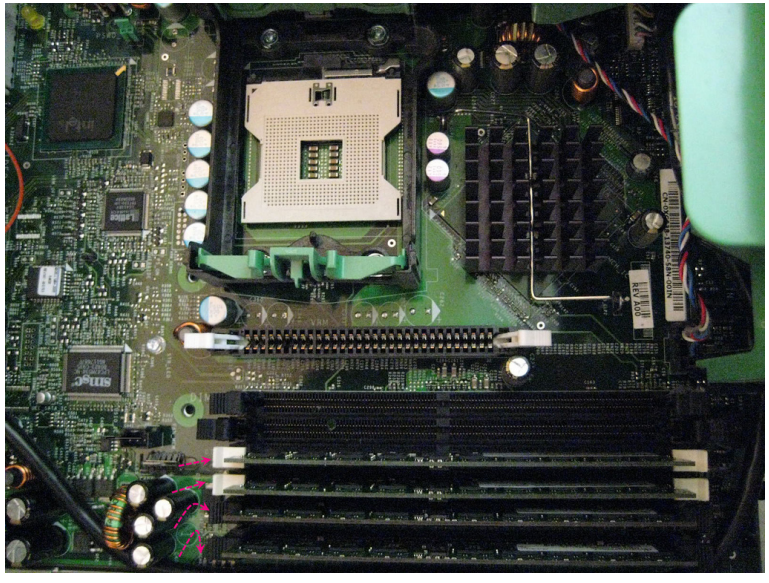
- ✓ For now, a *finite state machine*.

Our first Idealized Computer

A Finite State Machine has

- ✓ A finite set: possible **states** (configurations)
- ✓ Rules: how the system **transitions** between states
  - ▸ Depends on current state *and* current input
- ✓ Accept or reject, based on the final state

http://www.flickr.com/photos/alexbrn/5035170693



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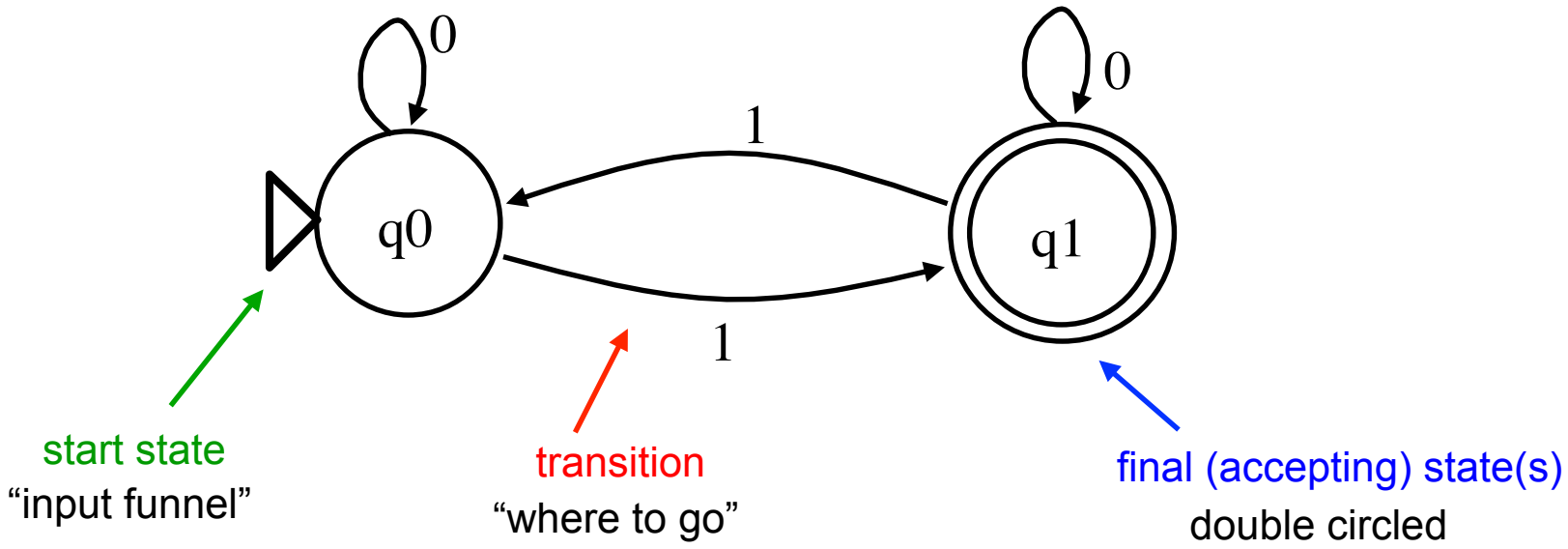
Deterministic Finite State Machines (DFAs)

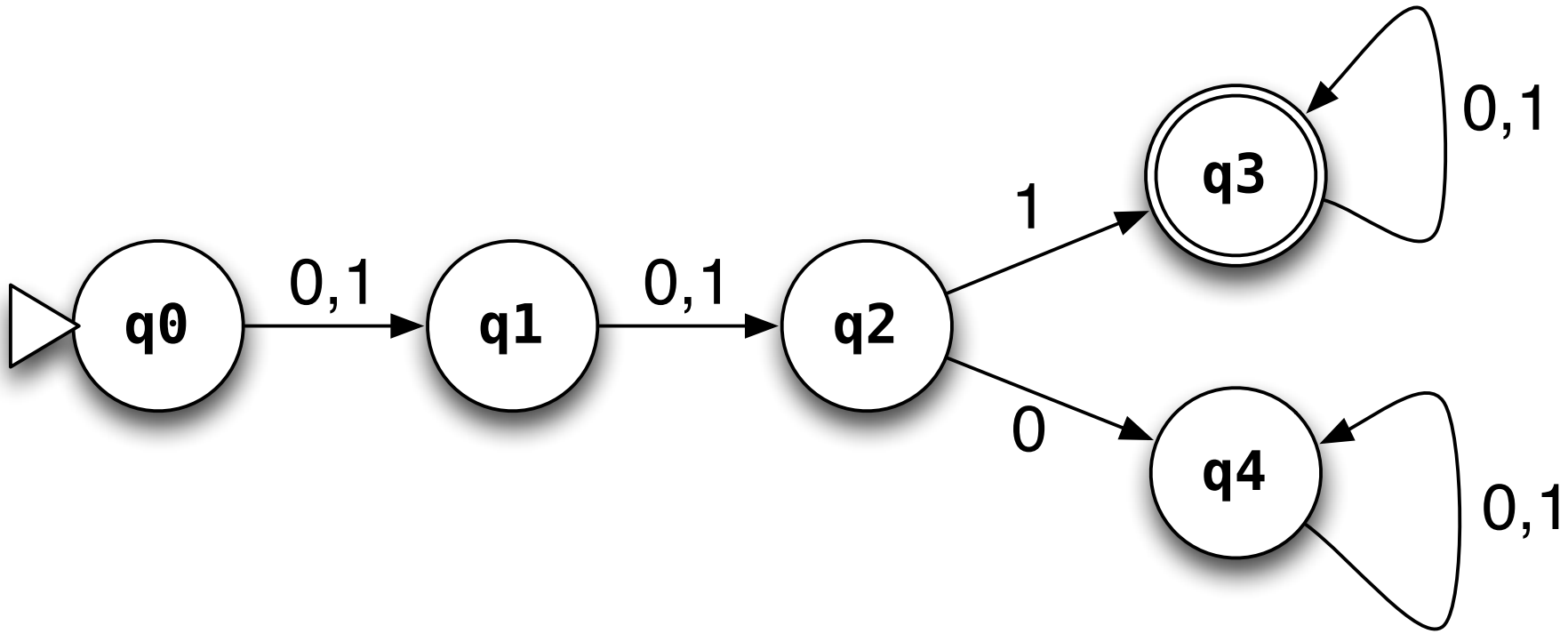
Every state has one transition for every possible input symbol.

Often represented graphically:



DFA Example

A DFA that accepts $L = \{ w \in \{0,1\}^* \mid w$'s **third** character is a 1 $\}$?

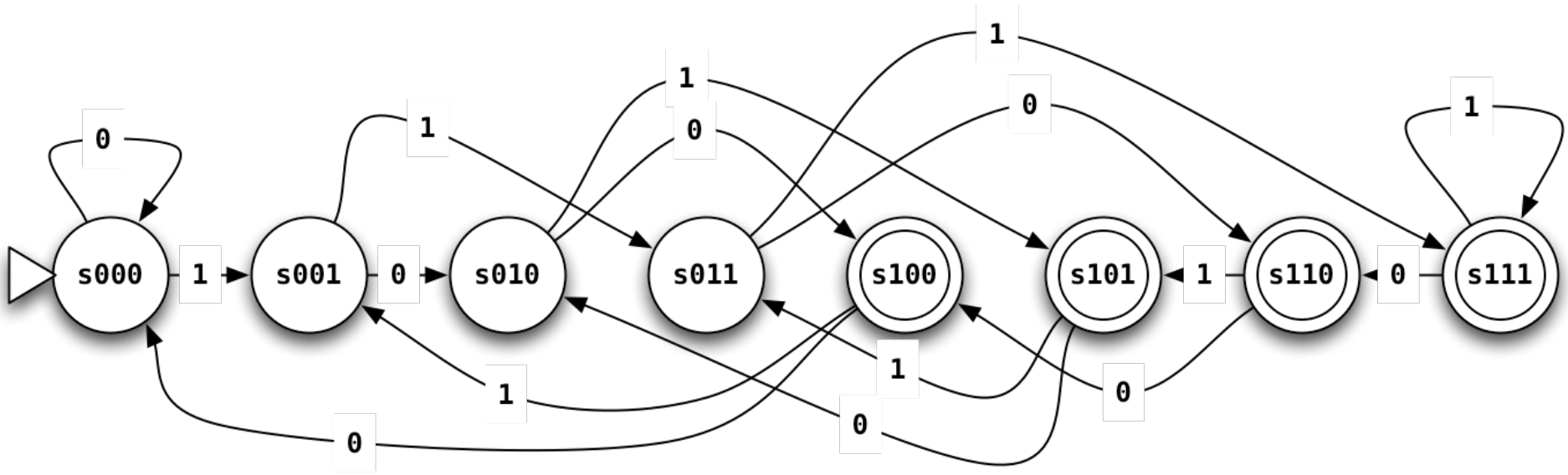


Does this DFA accept 010101? 011010?

Another DFA Example

A DFA that accepts

$L = \{ w \in \{0, 1\}^* \mid w$'s **third-to-last** character is a 1 $\}$?



Does this DFA accept 010101? 011010?

But Are Computers Really Deterministic?

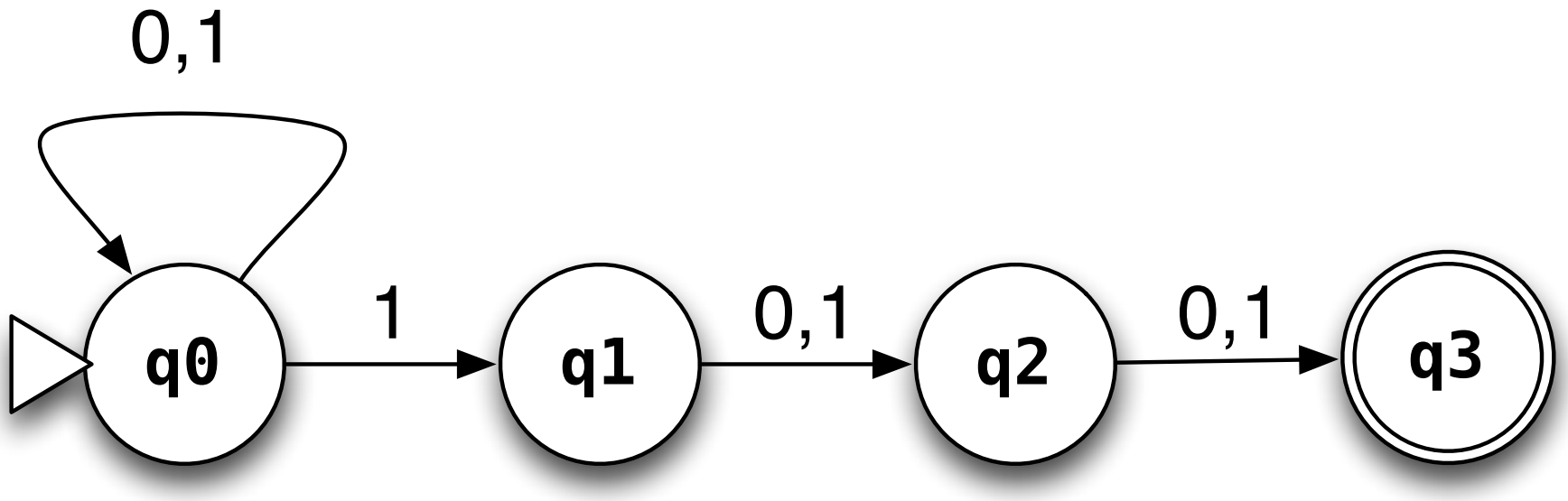
We plan to show that there are problems no DFA can solve.

But maybe *determinism* is the limiting factor?

DFAs + Nondeterminism = NFAs

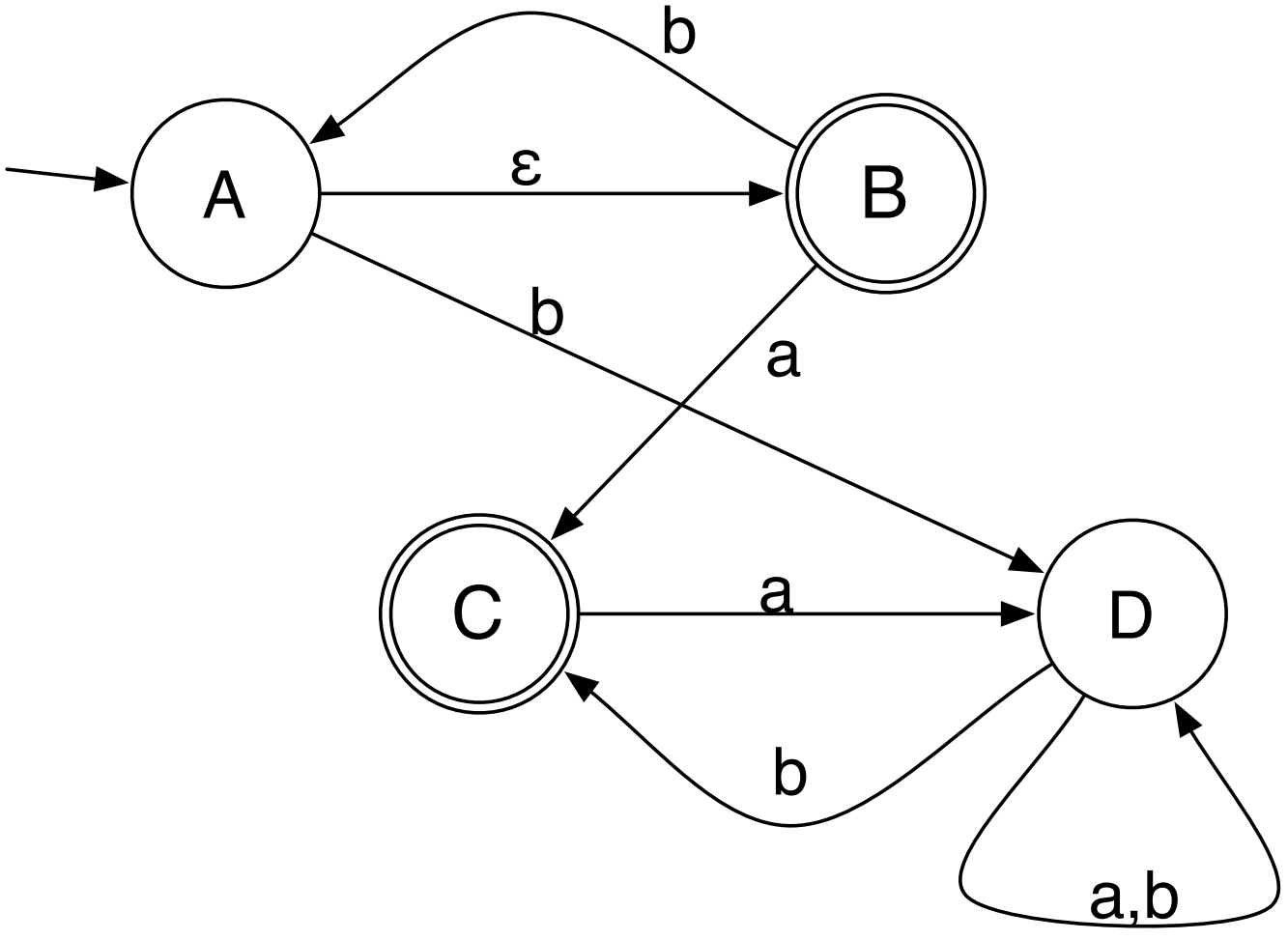
An NFA has the magic power to "guess" the right way to go.
It can even change states without reading input ("ε transitions").

Formally: NFA accepts iff **there is at least one accepting path.**



Does this NFA accept 010101? 011010?

What's Accepted? (bb? b? aaab? ba?)



A Jewel of Theoretical Computer Science



The following are equivalent:

1. There is a DFA accepting the set L.
2. There's an NFA accepting L [Rabin and Scott].
3. L can be described by a regular expression [Kleene].

In this case, L is called a **Regular Language**.

Regular Expressions do Pattern-Matching

$1^* \mid 1(00)^*$

$$\{ w \in \{0,1\}^* \mid w \text{ is a sequence of (zero or more) ones} \}$$
$$\cup$$
$$\{ w \in \{0,1\}^* \mid w \text{ is a } 1 \text{ followed by an even number of zeros.} \}$$

$b(ea|a)(r|d)$

$$\{\text{bear}, \text{bead}, \text{bar}, \text{bad}\}$$

$(0|1)(0|1)1(0|1)^*$

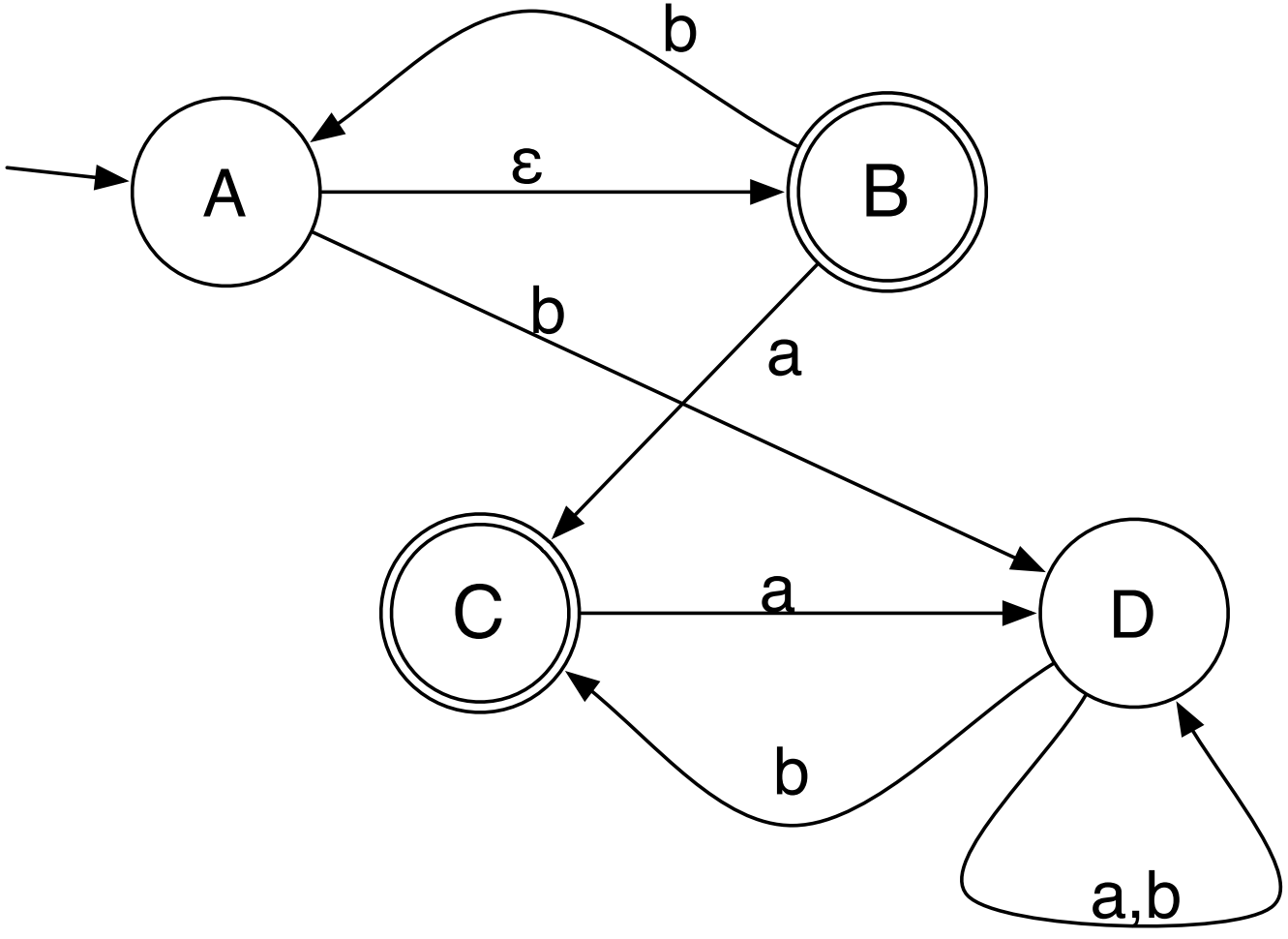
$$\{ w \in \{0,1\}^* \mid w\text{'s third character is a } 1 \}$$

Regular Expression Ingredients

Regular expressions are built from:

- ✓ $\emptyset$
- ✓ ε
- ✓ A single character from the alphabet (*e.g.*, $\mathtt{a}$ or $\mathtt{0}$)
- ✓ Concatenation: $r_1 r_2$
- ✓ Union: $r_1 \mid r_2$
- ✓ Repetition (Kleene Star): r_1^*.

Can this NFA be Simulated by a DFA?



Turn this Regular Expression into an NFA

$$1^* \mid 1(00)^*$$

CS 81 covers two ways to convert NFAs/DFAs into regular expressions

Practical Applications

Regular expressions are compact but powerful.

Often used to search/process text

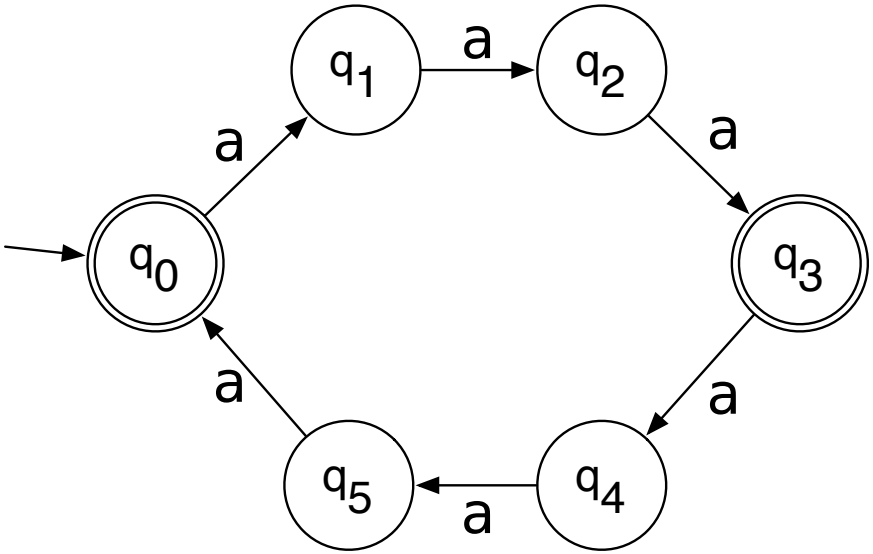
- ✓ E.g., `egrep`
- ✓ E.g., Assignment 9 (Unicalc Tokenizer)
- ✓ E.g., Almost *everything* written in `perl`

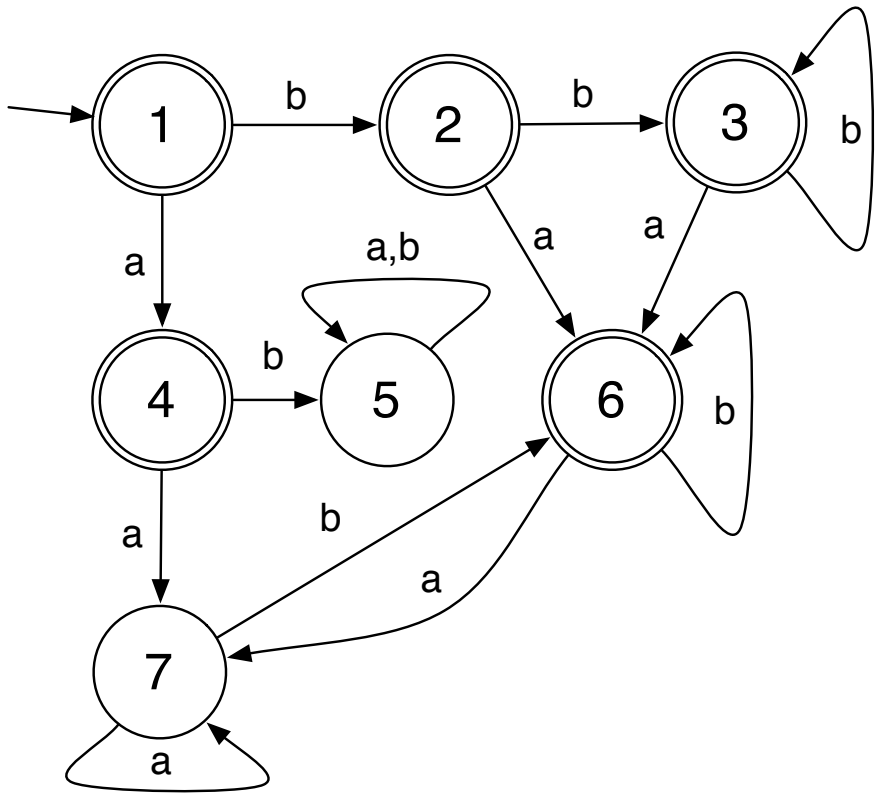
How to implement general regular-expression matching?

- ✓ Convert RE to a finite state machine
- ✓ Run input string(s) through!

DFA sizes

How can we know whether a DFA is as small as possible?





A DFA is minimal if every state is necessary (all states are distinguishable)

The Plan

1. Given languages, find *lower* bounds on the number of states in any corresponding DFA.
   - E.g., $\{ a^{3n} \mid n \geqslant 0 \}$ requires at least 3 states.
   - E.g., $\{ w \in \{0,1\}^* \mid w$'s **third-to-last** character is a 1 $\}$ requires at least 8 states.

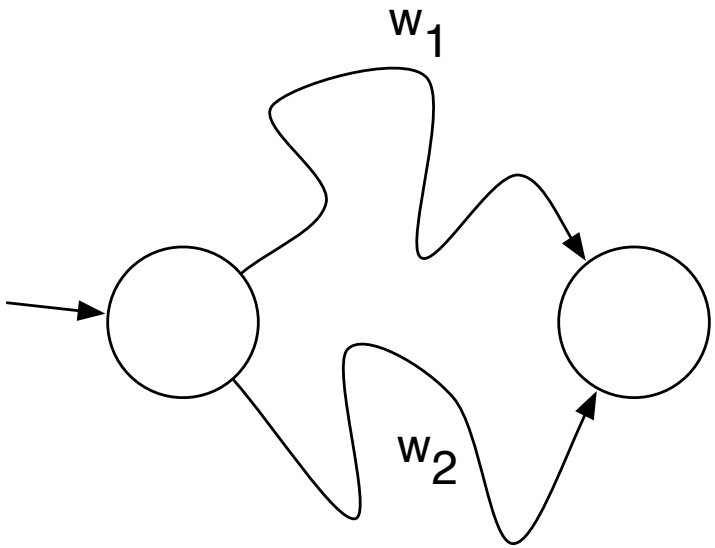
2. Find languages requiring $\geqslant \infty$ states (are accepted by no *finite* state machine).

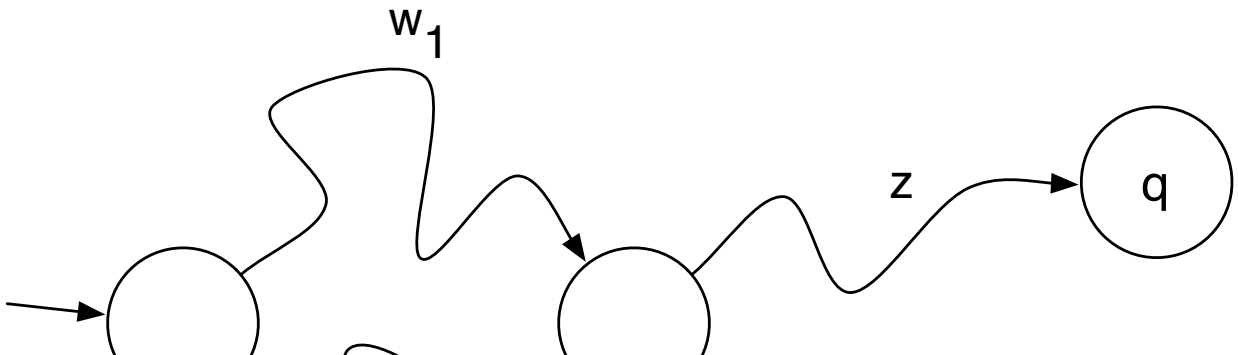
3. If computers are finite state machines, no computer can solve those problems!

State = Fate !

Suppose we have a state machine for some language.

Once two strings w_1 and w_2 lead to the same state, their future is indistiguishable.





State = Fate: The Contrapositive

If two strings lead to the same state, then the futures beyond this point are indistiguishable.

So, . . . if two strings in a language have distinguishable futures, they cannot lead to the same state.

Example: Suppose we have a state machine for

$$L = \{ a^{3n} \mid n \geqslant 0 \}.$$

Show that a, aa, and aaa must take us to three different states.

Corollary: any state machine for L has at least 3 states!

Distinguishability Theorem

Given a language L, strings w_1 and w_2 are **distinguishable** if

there exists $z \in \Sigma^*$ such that $w_1 z \in L$ and $w_2 z \notin L$ (or vice versa)

A set $S = \{w_1, w_2, \ldots\}$ is **pairwise distinguishable** if

w_i and w_j are distinguishable for all $i \neq j$.

Theorem

If there is a pairwise distinguishable set of N strings for a language L, then a DFA accepting L must have $\geqslant N$ states.

More Distinguishability Practice

$$L = \{ 0^i 1^j \mid i \text{ is even}, j \text{ is odd} \}$$

Are the following distinguishable?

1. 00 and 0
2. 0000 and 00
3. 010 and 1
4. ε and 001

Example

Prove that any DFA recognizing

$$L = \{ w \in \{0,1\}^* \mid w\text{'s third character is a } 1 \}$$

must have at least 5 states.

Proof: $\{ \varepsilon, 1, 0, 11, 10 \}$ Proof: $\{ \varepsilon, 0, 00, 000, 001 \}$

Theorem

If there is a pairwise distinguishable set of N states for a language L, then a DFA accepting L must have $\geq N$ states.

Nonregular Language Theorem

A language is said to be **regular** if it can be recognized by some DFA.

Consider the language

$$L = \{ 0^n 1^n \mid n \geqslant 0 \}$$

To show it's not regular, ...

Nonregular Language Theorem

A language is said to be **regular** if it can be recognized by some DFA.

Consider the language of correct multiplications of natural numbers:

$$L = \{ \quad 0\times0=0,\ \ 0\times1=0,\ \ \ldots \\ 1\times0=1,\ \ 1\times1=1,\ \ \ldots \\ 2\times0=1,\ \ 2\times1=2,\ \ \ldots \\ \}$$

What is the alphabet?

Prove that this language is not regular.

Conclusion

We can identify problems that no finite state machine can solve!

- ✓ Correct multiplications
- ✓ $L = \{ 0^n 1^n \mid n \geqslant 0 \}$
- ✓ $L = \{ w \mid w \text{ is a palindrome} \}$

If computers are finite state machines, no computer can solve these problems!

The End (?)

Moving On...

Any computer is a finite state machine.

Hence, there are questions it provably cannot answer. (E.g., "is the input a palindrome?")

But it sure seems like computers can answer these questions (until we run out of memory), so maybe this isn't so helpful in practice?

Perhaps we should consider more powerful models of computation. (But not too powerful! Why?)