Terminology For a formula φ, a term t and a variable x, we say *t is free (available) for x in φ* if in the parse tree T of φ, no variable y in $\mathrm{Free}(t)$ appears quantified along any leaf to root path for which the leaf is labeled by a variable x in $\mathrm{Free}(\varphi)$.

For a formula φ, a term t and a variable x, the notation $\varphi[t/x]$ denotes the formula obtained by substituting all occurrences of x in φ by t under the condition that t is free for x in φ.

Proof Rules

1. Conjunction Introduction ($\wedge\mathcal{I}$) and Eliminations ($\wedge\mathcal{E}_1$ and $\wedge\mathcal{E}_2$):

1	α
2	β
3	$(\alpha \wedge \beta)$

1	$(\alpha \wedge \beta)$
2	α

1	$(\alpha \wedge \beta)$
2	β

2. Disjunction Introductions ($\vee\mathcal{I}_1$ and $\vee\mathcal{I}_2$) and Elimination ($\vee\mathcal{E}$) (Proof by Cases)):

1	α
2	$(\alpha \vee \beta)$

1	α
2	$(\beta \vee \alpha)$

1	$(\alpha \vee \beta)$	
2		α
3		ω
4		β
5		ω
6	ω	

3. Implication Introduction ($\rightarrow\mathcal{I}$), Modus Ponens (MP), and Modus Tollens (MT):

1		α
2		β
3	$(\alpha \rightarrow \beta)$	

1	$(\alpha \rightarrow \beta)$
2	α
3	β

1	$(\alpha \rightarrow \beta)$
2	$\neg\beta$
3	$\neg\alpha$

4. Negation Introduction ($\neg\mathcal{I}$) (Proof By Contradiction) and Elimination ($\neg\mathcal{E}$):

1		α
2		$\bot$
3	$\neg\alpha$	

1	α
2	$\neg\alpha$
3	$\bot$

5. Falsehood Introduction ($\bot\mathcal{I}$) and Elimination ($\bot\mathcal{E}$):

$$\begin{array}{r|l} 1 & \alpha \\ 2 & \neg\alpha \\ \hline 3 & \bot \end{array} \qquad \begin{array}{r|l} 1 & \bot \\ \hline 2 & \alpha \end{array}$$

6. Double Negation Introduction ($\neg\neg\mathcal{I}$) and Elimination ($\neg\neg\mathcal{E}$):

$$\begin{array}{r|l} 1 & \alpha \\ \hline 2 & \neg\neg\alpha \end{array} \qquad \begin{array}{r|l} 1 & \neg\neg\alpha \\ \hline 2 & \alpha \end{array}$$

7. Law of Excluded Middle (LEM):

$$\begin{array}{r|l} 1 & (\alpha \vee \neg\alpha) \end{array}$$

8. Universal Introduction ($\forall\mathcal{I}$) and Elimination ($\forall\mathcal{E}$):

$$\begin{array}{r|l} 1 & \varphi[a/x] \\ \hline 2 & (\forall x)\varphi \end{array} \qquad \begin{array}{r|l} 1 & (\forall x)\varphi \\ \hline 2 & \varphi[t/x] \end{array}$$

Note: In $\forall\mathcal{I}$, a is any free variable/constant that does not appear in any non-discharged (unfinished) hypothesis (assumption) above the $\forall\mathcal{I}$ inference. In $\forall\mathcal{E}$, t is any term that is free for x in φ.

9. Existential Introduction ($\exists\mathcal{I}$) and Elimination ($\exists\mathcal{E}$):

$$\begin{array}{r|l} 1 & \varphi[t/x] \\ \hline 2 & (\exists x)\varphi \end{array} \qquad \begin{array}{r|l} 1 & (\exists x)\varphi \\ \hline 2 & \quad \begin{array}{|l} \varphi[a/x] \\ \hline \end{array} \\ 3 & \quad \begin{array}{|l} \omega \end{array} \\ 4 & \omega \end{array}$$

Note: In $\exists\mathcal{E}$, a is any free variable/constant that does not appear in any non-discharged (unfinished) hypotheses (assumption) other than $\varphi[a/x]$ above the hypothesis ω of the $\exists\mathcal{E}$ inference. In $\exists\mathcal{I}$, t is any term that is free for x in φ.

Example 1 Proving part of the Distributive Law: $(a \vee b) \wedge c \vdash (a \wedge c) \vee (b \wedge c)$.

1	$(a \vee b) \wedge c$	premise
2	$(a \vee b)$	$\wedge\mathcal{E}_1$, 1
3	$\quad a$	assumption
4	$\quad c$	$\wedge\mathcal{E}_2$, 1
5	$\quad (a \wedge c)$	$\wedge\mathcal{I}$, 3, 4
6	$\quad (a \wedge c) \vee (b \wedge c)$	$\vee\mathcal{I}_1$, 5
7	$\quad b$	assumption
8	$\quad c$	$\wedge\mathcal{E}_2$, 1
9	$\quad (b \wedge c)$	$\wedge\mathcal{I}$, 7, 8
10	$\quad (a \wedge c) \vee (b \wedge c)$	$\vee\mathcal{I}_2$, 8
11	$(a \wedge c) \vee (b \wedge c)$	$\vee\mathcal{E}$, 2, 3–6, 7–10

Example 2 Proving DeMorgan's Law for quantifiers: $\neg(\exists x)\neg A(x) \vdash (\forall x)A(x)$.

1	$\neg(\exists x)\neg A(x)$	premise
2	$\quad \neg A(a)$	assumption
3	$\quad (\exists x)\neg A(x)$	$\exists\mathcal{I}$, 2
4	$\quad \neg(\exists x)\neg A(x)$	Copy, 1
5	$\quad \bot$	$\neg\mathcal{E}$, 3, 4
6	$\neg\neg A(a)$	$\neg\mathcal{I}$, 2–5
7	$A(a)$	$\neg\neg\mathcal{E}$, 6
8	$(\forall x)A(x)$	$\forall\mathcal{I}$, 7

Note, $\forall\mathcal{I}$ is safe here since although a had appeared in an assumption $\neg A(a)$ earlier but this assumption was discharged (finished) when $\bot$ was derived.