

Gentzen introduced a proof procedure called the **sequent calculus** as an extension of his earlier natural deduction proof system. We follow a survey given by Buss [1].

In this method, each line in a proof is a *sequent* where a sequent is written in the form:

$$A_1, \dots, A_m \Rightarrow B_1, \dots, B_n$$

where the symbol \Rightarrow is a new symbol called the sequent arrow (not to be confused with the implication symbol \rightarrow) and where A_i and B_j are formulas. The intentional meaning of the above sequent is that the conjunction of the A_i 's implies the disjunction of the B_j 's. So, an equivalent way to write the above sequent is as the following implication:

$$\bigwedge_{i=1}^m A_i \rightarrow \bigvee_{j=1}^n B_j.$$

A standard convention for an empty conjunction ($m = 0$) is it evaluates to \top (True) while an empty disjunction ($n = 0$) evaluates to \perp (False). So, $\Rightarrow A \equiv A$, $A \Rightarrow \equiv \neg A$, and the empty sequent \Rightarrow is equivalent to \perp .

The sequence of formulas A_1, \dots, A_m is called the *antecedent* while the sequence of formulas B_1, \dots, B_n is called the *succedent*. Both of these sequences are called *cedents*.

To prove a formula α in sequent calculus, we build a proof tree (much like the tableaux method) by starting with a root labeled with α (at the bottom) and constructing the tree upwards. The leaves of the tree are called the initial sequents (axioms) which are logical axioms of the form $\beta \Rightarrow \beta$.

Inference Rules for Sequent Calculus

1. *Weakening* rules:

$$\text{weakening left: } \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}$$

$$\text{weakening right: } \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}$$

2. *Exchange* rules:

$$\text{exchange left: } \frac{\Gamma_1, A, B, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, B, A, \Gamma_2 \Rightarrow \Delta}$$

$$\text{exchange right: } \frac{\Gamma \Rightarrow \Delta_1, A, B, \Delta_2}{\Gamma \Rightarrow \Delta_1, B, A, \Delta_2}$$

3. *Contraction* rules:

$$\text{contraction left: } \frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}$$

$$\text{contraction right: } \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

4. *Negation Introduction* rules:

$$\text{negation left: } \frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}$$

$$\text{negation right: } \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}$$

5. *Conjunction Introduction* rules:

$$\text{conjunction left: } \frac{A, B, \Gamma \Rightarrow \Delta}{(A \wedge B), \Gamma \Rightarrow \Delta}$$

$$\text{conjunction right: } \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, (A \wedge B)}$$

6. *Disjunction Introduction* rules:

$$\text{disjunction left: } \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{(A \vee B), \Gamma \Rightarrow \Delta}$$

$$\text{disjunction right: } \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, (A \vee B)}$$

7. *Implication Introduction* rules:

$$\text{implication left: } \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{(A \rightarrow B), \Gamma \Rightarrow \Delta}$$

$$\text{implication right: } \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, (A \rightarrow B)}$$

8. *Cut* rule:

$$\text{cut: } \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

9. *Universal Introduction* rules:

$$\forall \text{ left: } \frac{A(t), \Gamma \Rightarrow \Delta}{(\forall x)A(x), \Gamma \Rightarrow \Delta}$$

$$\forall \text{ right: } \frac{\Gamma \Rightarrow \Delta, A(a)}{\Gamma \Rightarrow \Delta, (\forall x)A(x)}$$

10. *Existential Introduction* rules:

$$\exists \text{ left: } \frac{A(a), \Gamma \Rightarrow \Delta}{(\exists x)A(x), \Gamma \Rightarrow \Delta}$$

$$\exists \text{ right: } \frac{\Gamma \Rightarrow \Delta, A(t)}{\Gamma \Rightarrow \Delta, (\exists x)A(x)}$$

In the quantifier rules above, t is any term and a is any free variable which must not appear in Γ and Δ (in \forall -right and \exists -left).

Free and Bound Variables The following is an excerpt from Buss' survey [1]:

The first-order sequent calculus has two classes of variables, called *free variables* and *bound variables*. There are infinitely many variables of each type; free variables are denoted by metavariables a, b, c, \dots , and bound variables by the metavariables z, y, x, \dots . The essential idea is that free variables may not be quantified, while bound variables may not occur freely in formulas. The syntactic distinction between free and bound variables necessitates a change to the definition of terms and formulas. Firstly, *terms* are now defined as being built up from free variables and function symbols; whereas, the *semiterms* are defined as being built up from free and bound variables and function symbols. Secondly, only bound variables may ever be quantified. The set of *formulas* is now redefined with the additional requirement that only free variables may occur freely in formulas. *Semiformulas* are like formulas, except that bound variables may occur freely in semiformulas. We henceforth use r, s, t, \dots as metavariables for terms, and A, B, C, \dots as metavariables for formulas.

Example 1 Proving the Distributive Law:

$$(a \vee b) \wedge (a \vee c) \vdash a \vee (b \wedge c).$$

The proof (in complete form) of this law in sequent calculus is as follows:

$$\frac{\frac{\frac{a \Rightarrow a}{a \Rightarrow a, b} \quad \frac{\frac{b \Rightarrow b}{b \Rightarrow b, a}}{b \Rightarrow a, b}}{a \vee b \Rightarrow a, b} \quad \frac{\frac{\frac{a \Rightarrow a}{a \Rightarrow a, c} \quad \frac{\frac{c \Rightarrow c}{c \Rightarrow c, a}}{c \Rightarrow a, c}}{a \vee c \Rightarrow a, c}}{a \vee b, a \vee c \Rightarrow a, c}}{a \vee b, a \vee c \Rightarrow a, b \wedge c} \vee \text{right}}{a \vee b, a \vee c \Rightarrow a \vee (b \wedge c)} \wedge \text{left}}{(a \vee b) \wedge (a \vee c) \Rightarrow a \vee (b \wedge c)}$$

A more abbreviated form of the above proof is (with some implicit exchange steps):

$$\frac{\frac{\text{weakening} \frac{a \Rightarrow a}{a \Rightarrow a, b} \quad \text{weakening} \frac{b \Rightarrow b}{b \Rightarrow a, b}}{\vee \text{left} \frac{a \vee b \Rightarrow a, b}{a \vee b, a \vee c \Rightarrow a, b}} \quad \frac{\frac{a \Rightarrow a}{a \Rightarrow a, c} \quad \text{weakening} \frac{c \Rightarrow c}{c \Rightarrow a, c}}{a \vee c \Rightarrow a, c} \quad \text{weakening} \frac{a \vee b, a \vee c \Rightarrow a, c}{a \vee b, a \vee c \Rightarrow a, c}}{\wedge \text{right}}}{\frac{a \vee b, a \vee c \Rightarrow a, b \wedge c}{a \vee b, a \vee c \Rightarrow a \vee (b \wedge c)} \vee \text{right}} \wedge \text{left} \frac{a \vee b, a \vee c \Rightarrow a \vee (b \wedge c)}{(a \vee b) \wedge (a \vee c) \Rightarrow a \vee (b \wedge c)}$$

Example 2 Proving DeMorgan's Law: $\neg(\exists x)\neg A(x) \vdash (\forall x)A(x)$. The proof of this law in sequent calculus is as follows:

$$\neg \text{left} \frac{\frac{\frac{A(a) \Rightarrow A(a)}{A(a) \Rightarrow (\forall x)A(x)} \forall \text{right}}{\Rightarrow \neg A(a), (\forall x)A(x)} \neg \text{right}}{\Rightarrow (\exists x)\neg A(x), (\forall x)A(x)} \exists \text{right}}{\neg(\exists x)\neg A(x) \Rightarrow (\forall x)A(x)}$$

Example 3 Proving the sequent $(\forall x)A(x) \vee (\forall x)B(x) \vdash (\forall x)(A(x) \vee B(x))$.

$$\frac{\frac{\text{weakening} \frac{A(a) \Rightarrow A(a)}{A(a) \Rightarrow A(a), B(a)} \quad \frac{B(a) \Rightarrow B(a)}{B(a) \Rightarrow A(a), B(a)} \text{weakening}}{\vee \text{right} \frac{A(a) \Rightarrow A(a) \vee B(a)}{B(a) \Rightarrow A(a) \vee B(a)} \vee \text{right}} \quad \frac{\frac{A(a) \Rightarrow A(a)}{(\forall x)A(x) \Rightarrow A(a) \vee B(a)} \forall \text{left}}{\frac{B(a) \Rightarrow A(a) \vee B(a)}{(\forall x)B(x) \Rightarrow A(a) \vee B(a)} \forall \text{left}}}{\frac{(\forall x)A(x) \vee (\forall x)B(x) \Rightarrow A(a) \vee B(a)}{(\forall x)A(x) \vee (\forall x)B(x) \Rightarrow (\forall x)(A(x) \vee B(x))} \forall \text{right}}$$

References

- [1] Samuel R. Buss, "Introduction to Proof Theory," in *Handbook of Proof Theory*, S. Buss (editor), *Elsevier*, 1998, pages 1-78.