CS81 HW#3 Spring 2012 *Out: 02/01/2012. Due: 02/08/2012 (before class)*

Policy on homeworks

1. *Collaboration*: You may discuss a question with any other student currently taking CS81 provided: (i) both of you contribute equally; (ii) you come away from any discussion with an understanding in your mind (and no archived solution of any form is retained); (iii) your submission is your own work prepared by yourself on a separate occassion.
2. *Reference materials*: You should only refer to materials from this semester of CS81 (class notes, handouts, textbooks, grutors, instructor, etc).
3. *Submission*: Your submission should be legible or is prepared using TeX.

Prove or Disprove If **prove**, use only the Natural Deduction rules given below (augmented with the Copy rule). If **disprove**, provide a *model* where the assertion does not hold. Some of the problems are adapted from Huth and Ryan.

Convention All predicates are denoted by upper-case letters. All variables are denoted by lower-case letters late in the alphabet (such as x, y, z). All atomic constants are denoted by lower-case letters early in the alphabet (such as a, b, c).

1. Negating an Existential Negative means Universal Positive:

1	$\neg(\exists x)\neg P(x)$	premise
2	$(\forall x)P(x)$	goal

2. One of the Distributive Laws for Quantifiers:

1	$(\forall x)[P(x) \vee Q(x)]$	premise
2	$(\forall x)P(x) \vee (\forall x)Q(x)$	goal

3. The Other Part of the Distributive Laws for Quantifiers:

1	$(\exists x)[P(x) \wedge Q(x)]$	premise
2	$(\exists x)P(x) \wedge (\exists x)Q(x)$	goal

4. Irreflexive and Transitive means Never Having to Be Symmetric:

1	$(\forall x)(\forall y)(\forall z)[(E(x,y) \wedge E(y,z)) \rightarrow E(x,z)]$	premise
2	$(\forall x)\neg E(x,x)$	premise
3	$(\forall x)(\forall y)[E(x,y) \rightarrow \neg E(y,x)]$	goal

5. Universal Modus Ponens:

1	$(\forall x)[P(x) \rightarrow (Q(x) \vee R(x))]$	premise
2	$\neg(\exists x)[P(x) \wedge R(x)]$	premise
3	$(\forall x)[P(x) \rightarrow Q(x)]$	goal

6. Triple Substitution:

1	$(\forall x)[P(x) \rightarrow (\forall y)E(y,y,\hat{b})]$	premise
2	$(\forall x)(\forall y)[E(\hat{a},y,x) \rightarrow F(x,y,y)]$	premise
3	$P(\hat{a}) \rightarrow (\exists x)F(x,\hat{c},\hat{c})$	goal

7. Law of Included Middle:

1	$(\forall x)(\exists y)L(x,y)$	premise
2	$(\forall x)(\forall y)[L(x,y) \rightarrow (\exists z)(L(x,z) \wedge L(z,y))]$	goal

8. Existential Tag:

1	$(\exists x)[P(x) \vee (\forall y)(E(x,y) \rightarrow P(y))]$	premise
2	$(\forall x)[E(x,\hat{a}) \wedge \neg P(\hat{a})]$	premise
3	$(\exists x)P(x)$	goal

9. Quantified Proof by Cases:

1	$(\forall x)(\forall y)[(B(y) \vee C(x)) \rightarrow Z(y)]$	premise
2	$(\forall x)(\exists y)[A(x) \rightarrow Z(y)]$	premise
3	$(\exists x)[A(x) \vee B(x)]$	premise
4	$(\exists x)Z(x)$	goal

10. Existential Modus Tollens:

1	$(\forall x)[A(x) \to Z(x)]$	premise
2	$(\exists x)[B(x) \land Z(x)] \land (\exists x)[B(x) \land \neg Z(x)]$	premise
3	$\neg(\forall x)[B(x) \to A(x)]$	goal

11. Reducibility:

1	$(\forall x)(\forall y)[(E(x,\hat{a}) \land E(\hat{a},y)) \to E(x,y)]$	premise
2	$(\forall x)[P(x) \to E(\hat{a},x)]$	premise
3	$(\exists x)[P(x) \land E(x,\hat{a})]$	premise
4	$(\exists x)(P(x) \land (\forall y)[P(y) \to E(x,y)])$	goal

12. Contrapositive Law for Quantifiers:

1	$(\forall x)(\forall y)(A(x) \to B(y))$	premise
2	$(\exists y)[\neg B(y)] \to (\forall x)[\neg A(x)]$	goal

13. Contrapositive Law Reversed:

1	$(\exists y)[\neg B(y)] \to (\forall x)[\neg A(x)]$	premise
2	$(\forall x)(\forall y)(A(x) \to B(y))$	goal

14. Yet Another Contrapositive Law for Quantifiers:

1	$(\exists x)(\forall y)(A(x) \to B(y))$	premise
2	$\neg(\forall x)A(x) \lor (\forall y)B(y)$	goal

15. Existential Loops:

1	$(\forall x)(\forall y)[(P(x) \land P(y)) \to (E(x,y) \to E(y,x))]$	premise
2	$(\forall x)(\forall y)(\forall z)\{[(P(x) \land P(y)) \land P(z)] \to [(E(x,y) \land E(y,z)) \to E(x,z)]\}$	premise
3	$(\forall x)([P(x) \land (\exists z)(P(z) \land E(x,z))] \to E(x,x))$	goal

Natural Deduction Rules for Predicate Logic

1. Conjunction Introduction $\wedge\mathcal{I}$ and Conjunction Eliminations $\wedge\mathcal{E}1$ and $\wedge\mathcal{E}2$:

$$\begin{array}{r|l} 1 & \alpha \\ 2 & \beta \\ \hline 3 & (\alpha \wedge \beta) \end{array} \qquad \begin{array}{r|l} 1 & (\alpha \wedge \beta) \\ \hline 2 & \alpha \end{array} \qquad \begin{array}{r|l} 1 & (\alpha \wedge \beta) \\ \hline 2 & \beta \end{array}$$

2. Disjunction Introductions $\vee\mathcal{I}1$ and $\vee\mathcal{I}2$, and Disjunction Elimination $\vee\mathcal{E}$ (Proof by Cases):

$$\begin{array}{r|l} 1 & \alpha \\ \hline 2 & (\alpha \vee \beta) \end{array} \qquad \begin{array}{r|l} 1 & \alpha \\ \hline 2 & (\beta \vee \alpha) \end{array} \qquad \begin{array}{r|l} 1 & (\alpha \vee \beta) \\ 2 & \quad \begin{array}{|l} \alpha \\ \hline \gamma \end{array} \\ 3 & \\ 4 & \quad \begin{array}{|l} \beta \\ \hline \gamma \end{array} \\ 5 & \\ 6 & \gamma \end{array}$$

3. Implication Introduction $\rightarrow\mathcal{I}$ and Implication Eliminations (Modus Ponens (MP) and Modus Tollens (MT)):

$$\begin{array}{r|l} 1 & \quad \begin{array}{|l} \alpha \\ \hline \beta \end{array} \\ 2 & \\ 3 & (\alpha \rightarrow \beta) \end{array} \qquad \begin{array}{r|l} 1 & (\alpha \rightarrow \beta) \\ 2 & \alpha \\ \hline 3 & \beta \end{array} \qquad \begin{array}{r|l} 1 & (\alpha \rightarrow \beta) \\ 2 & \neg\beta \\ \hline 3 & \neg\alpha \end{array}$$

4. Negation Introduction $\neg\mathcal{I}$ (Proof By Contradiction) and Negation Elimination $\neg\mathcal{E}$:

$$\begin{array}{r|l} 1 & \quad \begin{array}{|l} \alpha \\ \hline \bot \end{array} \\ 2 & \\ 3 & \neg\alpha \end{array} \qquad \begin{array}{r|l} 1 & \alpha \\ 2 & \neg\alpha \\ \hline 3 & \bot \end{array}$$

5. Falsehood Introduction $\bot\mathcal{I}$ and Falsehood Elimination $\bot\mathcal{E}$:

$$\begin{array}{r|l} 1 & \alpha \\ 2 & \neg\alpha \\ \hline 3 & \bot \end{array} \qquad \begin{array}{r|l} 1 & \bot \\ \hline 2 & \alpha \end{array}$$

6. Double Negation Introduction $\neg\neg\mathcal{I}$ and Double Negation Elimination $\neg\neg\mathcal{E}$:

$$\begin{array}{r|l} 1 & \alpha \\ \hline 2 & \neg\neg\alpha \end{array} \qquad \begin{array}{r|l} 1 & \neg\neg\alpha \\ \hline 2 & \alpha \end{array}$$

7. Law of Excluded Middle (LEM):

$$\begin{array}{r|l} 1 & (\alpha \vee \neg\alpha) \end{array}$$

8. Universal Introduction $\forall\mathcal{I}$ and Universal Elimination $\forall\mathcal{E}$:

$$\begin{array}{r|l} 1 & \alpha(a) \\ \hline 2 & (\forall x)\alpha(x) \end{array} \qquad \begin{array}{r|l} 1 & (\forall x)\alpha(x) \\ \hline 2 & \alpha(a) \end{array}$$

Here a stands for an arbitrary (free) constant.

9. Existential Introduction $\exists\mathcal{I}$ and Existential Elimination $\exists\mathcal{E}$:

$$\begin{array}{r|l} 1 & \alpha(a) \\ \hline 2 & (\exists x)\alpha(x) \end{array} \qquad \begin{array}{r|l} 1 & (\exists x)\alpha(x) \\ \hline 2 & \alpha(\hat{a}) \end{array}$$

Here a stands for some constant (free or otherwise) and $\hat{a}$ stands for a bound (not free) constant.

Example Prove or disprove this quarter of DeMorgan's Law for Quantifiers:

$$(\forall x)[\neg P(x)] \vdash \neg(\exists x)P(x).$$

Prove! Here is the derivation using Natural Deduction rules:

1	$(\forall x)[\neg P(x)]$	premise
2	$\quad (\exists x)P(x)$	assumption
3	$\quad P(\hat{a})$	$\exists\mathcal{E}$, 2
4	$\quad \neg P(\hat{a})$	$\forall\mathcal{E}$, 1
5	$\quad \bot$	$\neg\mathcal{E}$, 3, 4
6	$\neg(\exists x)P(x)$	$\neg\mathcal{I}$, 2–5