CS81 HW#4 Spring 2012 *Out: 02/08/2012. Due: 02/15/2012 (before class)*

Policy on homeworks

1. *Collaboration*: You may discuss a question with any other student currently taking CS81 provided: (i) both of you contribute equally; (ii) you come away from any discussion with an understanding in your mind (and no archived solution of any form is retained); (iii) your submission is your own work prepared by yourself on a separate occassion.

2. *Reference materials*: You should only refer to materials from this semester of CS81 (class notes, handouts, textbooks, grutors, instructor, etc).

3. *Submission*: Your submission should be legible or is prepared using TeX.

Logical Problems

1. Let L_S be a first-order language with one constant e and one binary function f. So $S = \{e^{(0)}, f^{(2)}\}$. Consider the following first-order sentences over L_S.

$$
\begin{aligned}
\phi_1 &= (\forall x)(\forall y)(\forall z)[f(x, f(y,z)) = f(f(x,y),z)] \\
\phi_2 &= (\forall x)[f(x,e) = x] \\
\phi_3 &= (\forall x)(\exists y)[f(x,y) = e] \\
\phi_4 &= (\forall x)(\exists y)[f(y,x) = e] \\
\phi_5 &= (\forall x)[(\forall y)(f(y,x) = y) \to (x = e)] \\
\phi_6 &= (\forall x)(\forall y)(\forall z)[(f(x,z) = e) \to (f(f(x,y),z) = y)] \\
\phi_7 &= (\forall x)(\forall y)(\forall z)[((f(x,y) = e) \wedge (f(x,z) = e)) \to (y = z)]
\end{aligned}
$$

   Let $\Phi = \{\phi_1, \phi_2, \phi_3\}$. Prove or disprove the following semantic entailments:

   (a) $\Phi \models \phi_4$.

   (b) $\Phi \models \phi_5$.

   (c) $\Phi \models \phi_6$.

   (d) $\Phi \models \phi_7$.

   You may need to invoke some rules on equality.

2. Imagine Waldo playing a game on a Boolean circular array of size N.

   (a) Each cell of the array contains either $\perp$ or $\top$.

   (b) Waldo's state of mind can either be $\mathrm{pessimistic}$, $\mathrm{ambivalent}$, or $\mathrm{optimistic}$.

   (c) Waldo's initial position on the array is at location 0, but he is allowed to move after each step to his left or right. If he moves left from 0 then he will arrive at location $N-1$. Similarly, moving right off $N-1$ will get him to 0.

(d) At each step of the game, Waldo decides based on his current state of mind (say q) and the contents of the array cell at his current location (say a), what his state of mind will be on the next step (say q'), what symbol to write to the current location of the array (say b), and whether to move left or right within the circular array (say d). All this can be encoded as a table of mood transitions of the form $(q, a) \mapsto (q', b, d)$:

$$\begin{array}{rcl} (\text{pessimistic}, \bot) & \mapsto & (\text{pessimistic}, \bot, \text{left}) \\ (\text{ambivalent}, \bot) & \mapsto & (\text{pessimistic}, \top, \text{left}) \\ (\text{optimistic}, \bot) & \mapsto & (\text{ambivalent}, \bot, \text{right}) \\ (\text{pessimistic}, \top) & \mapsto & (\text{ambivalent}, \bot, \text{left}) \\ (\text{ambivalent}, \top) & \mapsto & (\text{optimistic}, \bot, \text{right}) \\ (\text{optimistic}, \top) & \mapsto & (\text{optimistic}, \top, \text{right}) \end{array}$$

(e) Waldo's initial state of mind is pessimistic.

(f) Waldo plays T steps of this game.

For example, after $N = 5$ steps on input $[\top, \top, \top]$, Waldo ends up at 0 feeling ambivalent and leaving the array as $[\bot, \bot, \bot]$. Consider the following propositional atoms:

- $S(t, q) = \top$ if and only if at step t Waldo's state of mind is q.
- $L(t, j) = \top$ if and only if at step t Waldo's location is j.
- $M(t, j, b) = \top$ if and only if at step t the contents of array cell j is b.

Here $0 \leq t \leq T$, $q \in \{\text{pessimistic}, \text{ambivalent}, \text{optimistic}\}$, $0 \leq j < N$, and $b \in \{\bot, \top\}$.

Express each of the following conditions as propositional formulas:

C1 At each step t, Waldo is in exactly one state of mind.

C2 At each step t, Waldo is located at exactly one array location.

C3 At each step t, each cell of the array contains exactly one Boolean value.

C4 The initial condition at step 0 is that Waldo is pessimistic and is at location 0, and the contents of the array is specified by a set of variables $x_0, \ldots, x_{N-1}$.

C5 The states of the game between steps t and $t + 1$ obeys the transition table given above.

C6 The final condition at step T is that Waldo is optimistic and is back at location 0.

As an example, for C4 we should have the formula:

$$\phi_4 = S(0, \text{pessimistic}) \wedge L(0, 0) \wedge \bigwedge_{j=0}^{N-1} M(0, j, x_j)$$

State your formulas in *clausal* form (a conjunctive collection of disjunctions).