CS81 HW#5 Spring 2012 *Out: 02/15/2012. Due: 02/22/2012 (before class)*

Policy on homeworks

1. *Collaboration*: You may discuss a question with any other student currently taking CS81 provided: (i) both of you contribute equally; (ii) you come away from any discussion with an understanding in your mind (and no archived solution of any form is retained); (iii) your submission is your own work prepared by yourself on a separate occassion.
2. *Reference materials*: You should only refer to materials from this semester of CS81 (class notes, handouts, textbooks, grutors, instructor, etc).
3. *Submission*: Your submission should be legible or is prepared using TeX.

Gödel's peak via Henkin's trail[1]

Fix a first-order language L_S over a **countable** alphabet S. Let Φ be a collection of formulas of L_S. Assume the **Soundness** Theorem for ND holds: if $\Phi \vdash \varphi$, then $\Phi \models \varphi$, for any formula φ. Let $\mathrm{Free}(\Phi)$ denote the set of free (unbound) variables which appear in Φ.

PROVE CAREFULLY ALL THE CLAIMS IN THE SHADOWED BOXES.

I. (a) We say Φ is **consistent** (*notation*: $\mathrm{Cons}(\Phi)$) iff there is no formula ϕ so that $\Phi \vdash \phi$ and $\Phi \vdash \neg\phi$.

(b) We say Φ is **satisfiable** iff there is a model $\mathfrak{M}$ for which $\mathfrak{M} \models \Phi$.

II. Φ is not consistent iff $\Phi \vdash \varphi$, for any formula φ.

Proof. ($\Rightarrow$) If Φ is not consistent, then there is a formula ψ for which $\Phi \vdash \psi$ and $\Phi \vdash \neg\psi$. From this we obtain $\Phi \vdash \bot$ and thus $\Phi \vdash \varphi$, for any formula φ.
($\Leftarrow$) If $\Phi \vdash \psi$, for any formula ψ, then $\Phi \vdash \varphi$ and $\Phi \vdash \neg\varphi$, for any formula φ. Thus, Φ is not consistent. □

III. If Φ is satisfiable, then Φ is consistent.

Proof. If Φ is not consistent, then for some formula φ we have $\Phi \vdash \varphi$ and $\Phi \vdash \neg\varphi$. By the Soundness Theorem, $\Phi \models \varphi$ and $\Phi \models \neg\varphi$. So, Φ is not satisfiable since no model can satisfy both φ and $\neg\varphi$. □

The *structure* of the above proof (in Natural Deduction *itself*) is:

1	Φ is not consistent	assumption
2	$(\exists\varphi)[(\Phi \vdash \varphi) \wedge (\Phi \vdash \neg\varphi)]$	definition of inconsistent, 1
3	$(\Phi \vdash \hat{\varphi}) \wedge (\Phi \vdash \neg\hat{\varphi})$	$\exists\mathcal{E}$, 2
4	$\Phi \vdash \hat{\varphi}$	$\wedge\mathcal{E}$, 3
5	$\Phi \models \hat{\varphi}$	by Soundness Theorem, 4

[1]Map by Ebbinghaus, Flum and Thomas.

6	$\Phi \vdash \neg\hat{\varphi}$	$\wedge\mathcal{E}$, 3
7	$\Phi \models \neg\hat{\varphi}$	by Soundness Theorem, 6
8	$\Phi \models (\hat{\varphi} \wedge \neg\hat{\varphi})$	by property of semantic entailment
9	$\Phi \models \bot$	identity for $\bot$
10	Φ is not satisfiable	since no model can satisfy $\bot$

IV. For each formula φ, we have:

(a) If $\Phi \nvdash \varphi$, then $\mathrm{Cons}(\Phi \cup \{\neg\varphi\})$.

Proof. Suppose that $\Phi \cup \{\neg\varphi\}$ is not consistent. Then, for some formula ψ, we have $\Phi \cup \{\neg\varphi\} \vdash \psi$ and $\Phi \cup \{\neg\varphi\} \vdash \neg\psi$. Thus, we may deduce $\Phi \vdash \varphi$ as follows:

1	Φ		premise
2		$\neg\varphi$	assumption
3		ψ	since $\Phi, \neg\varphi \vdash \psi$
4		$\neg\psi$	since $\Phi, \neg\varphi \vdash \neg\psi$
5		$\bot$	$\neg\mathcal{I}$, 3, 4
6	$\neg\neg\varphi$		$\neg\mathcal{I}$, 2–5
7	φ		$\neg\neg\mathcal{E}$, 6

$\square$

(b) If $\mathrm{Cons}(\Phi)$ and $\Phi \vdash \varphi$, then $\mathrm{Cons}(\Phi \cup \{\varphi\})$.

(c) If $\mathrm{Cons}(\Phi)$, then $\mathrm{Cons}(\Phi \cup \{\varphi\})$ or $\mathrm{Cons}(\Phi \cup \{\neg\varphi\})$.

V. Φ is consistent iff any finite subset of Φ is consistent.

VI. (a) We say Φ is **maximally consistent** iff $\mathrm{Cons}(\Phi)$ and $\forall\varphi$ if $\mathrm{Cons}(\Phi \cup \{\varphi\})$ then $\varphi \in \Phi$.

(b) We say Φ has **witnesses** iff $\forall\varphi = (\exists x)\psi$, there is a term t where $((\exists x)\psi \to \psi[t/x]) \in \Phi$.

VII. Let Φ be maximally consistent and has witnesses. Then, for any formulas φ and ψ:

(a) If $\Phi \vdash \varphi$ then $\varphi \in \Phi$.

(b) Either $\varphi \in \Phi$ or $\neg\varphi \in \Phi$.

(c) $(\varphi \vee \psi) \in \Phi$ iff $\varphi \in \Phi$ or $\psi \in \Phi$.

(d) If $(\varphi \to \psi) \in \Phi$ and $\varphi \in \Phi$, then $\psi \in \Phi$.

(e) $(\exists x)\psi \in \Phi$ iff there is a term t with $\psi[t/x] \in \Phi$.

VIII. Let T_S be the set of all S-terms in L_S. We define an **equivalence relation** $\sim$ on T_S as:

$$t_0 \sim t_1 \quad \text{iff} \quad (t_0 = t_1) \in \Phi.$$

For a term t, let $[t]$ be the equivalence class of t which is the set of terms equivalent to t:

$$[t] = \{t' \in T_S : t \sim t'\}.$$

Let T_Φ be the set of equivalence classes:

$$T_\Phi = \{[t] : t \in T_S\}.$$

IX. Let $\mathfrak{A}_\Phi = (T_\Phi, \mathfrak{a})$ be an S-**structure** defined as follows:

(a) For each constant $c \in S$, we have $\mathfrak{a}(c) = [c]$.
(b) For each n-ary function symbol $f \in S$, we have $\mathfrak{a}(f)([t_0], \ldots, [t_{n-1}]) = [f(t_0, \ldots, t_{n-1})]$.
(c) For each n-ary relation symbol $R \in S$:

$$([t_0], \ldots, [t_{n-1}]) \in \mathfrak{a}(R) \quad \text{iff} \quad R(t_0, \ldots, t_{n-1}) \in \Phi.$$

Let σ_Φ be a valuation of the variables where for each variable $x \in S$, we have $\sigma_\Phi(x) = [x]$.
This defines a **model** $\mathfrak{M}_\Phi = (\mathfrak{A}_\Phi, \sigma_\Phi)$ over the language L_S.

X. (a) For all terms t, we have $\mathfrak{M}_\Phi(t) = [t]$.
(b) For all *atomic* formulas φ, we have $\mathfrak{M}_\Phi \models \varphi$ iff $\varphi \in \Phi$.

XI. Let Φ be maximally consistent and has witnesses. Then, for each formula φ, $\mathfrak{M}_\Phi \models \varphi$ iff $\varphi \in \Phi$.

XII. Let Φ be consistent and $\text{Free}(\Phi)$ is **finite**. Then, there is a consistent $\Psi \supseteq \Phi$ which has *witnesses*.

XIII. Let Ψ be consistent. Then, there is a *maximally consistent* $\Theta \supseteq \Psi$.

XIV. If Φ is consistent, then Φ satisfiable.

XV. If $\Phi \models \varphi$, then $\Phi \vdash \varphi$.