CS81 Spring 2012 NOTES ON MISPROOFS

This is a collection of some subtle but perhaps common missteps in proofs within Natural Deduction. We follow Huth and Ryan's (page 113) treatment of $\exists\mathcal{E}$ as a disjunction elimination. This interpretation has the benefit of forcing the instantiation into an explicit assumption (by bringing anything else attached to that assumption along).

Remark: These notes are perpetually under construction. Consider contributing examples which you think are constructive. Please consult the revised set of rules on Natural Deduction or Buss' survey chapter on First-Order Logic.

1. **Generalization**:

1	$(\forall x)[P(x) \vee Q(x)]$	premise
2	$P(a) \vee Q(a)$	$\forall\mathcal{E}$, 1
3	$\quad P(a)$	assumption
4	$\quad (\forall x)P(x)$	$\forall\mathcal{I}$ **mistake**, 3
5	$\quad (\forall x)P(x) \vee (\forall x)Q(x)$	$\vee\mathcal{I}_1$, 4
6	$\quad Q(a)$	assumption
7	$\quad (\forall x)Q(x)$	$\forall\mathcal{I}$ **mistake**, 6
8	$\quad (\forall x)P(x) \vee (\forall x)Q(x)$	$\vee\mathcal{I}_2$, 7
9	$(\forall x)P(x) \vee (\forall x)Q(x)$	$\vee\mathcal{E}$, 2, 3–5, 6–8

The mistake is that a is not free when $\forall\mathcal{I}$ is applied since it is involved in an unfinished assumption $P(a)$ (and $Q(a)$).

Contrast this with the following derivation

1	$(\forall x)(\exists y)[P(x) \rightarrow Q(y)]$	premise
2	$P(a)$	assumption
3	$(\exists y)[P(a) \rightarrow Q(y)]$	$\forall\mathcal{E}$, 1
4	$P(a) \rightarrow Q(\hat{b})$	$\exists\mathcal{E}$, 2
5	$Q(\hat{b})$	Modus Ponens, 2, 4
6	$(\exists y)Q(y)$	$\exists\mathcal{I}$, 5
7	$(\exists y)Q(y)$	$\exists\mathcal{E}$, 3
8	$P(a) \rightarrow (\exists y)Q(y)$	$\Rightarrow \mathcal{I}$, 2–7
9	$(\forall x)[P(x) \rightarrow (\exists y)Q(y)]$	$\forall\mathcal{I}$, 8

Here, applying $\forall\mathcal{I}$ on a on line 8 is valid since the assumption $P(a)$ on line 2 was already discharged.

2. **Elimination**:

1	$(\exists x)P(x) \wedge (\exists x)Q(x)$	premise
2	$(\exists x)P(x)$	$\wedge\mathcal{E}_1$, 1
3	$P(\hat{a})$	assumption
4	$(\exists x)Q(x)$	$\wedge\mathcal{E}_2$, 1
5	$Q(\hat{a})$	$\exists\mathcal{E}$ **mistake**: $\hat{a}$ is not free
6	$P(\hat{a}) \wedge Q(\hat{a})$	$\wedge\mathcal{I}$, 3, 5
7	$(\exists x)[P(x) \wedge Q(x)]$	$\exists\mathcal{I}$, 6
8	$(\exists x)[P(x) \wedge Q(x)]$	$\exists\mathcal{E}$, 4, 5–7
9	$(\exists x)[P(x) \wedge Q(x)]$	$\exists\mathcal{E}$, 2, 3–8

This mistake appears since $\hat{a}$ (on line 5) is involved in an unfinished assumption (on line 3).

3. **Ordering**:

1	$(\forall x)(\exists y)[P(x) \rightarrow Q(y)]$	premise
2	$(\forall x)[P(x) \rightarrow Q(\hat{a})]$	$\exists\mathcal{E}$ **mistake**: ordering
3	$(\exists y)(\forall x)[P(x) \rightarrow Q(y)]$	$\exists\mathcal{I}$, 2

The mistake is that $(\exists y)$ cannot be eliminated before $(\forall x)$ is eliminated.

4. **Scoping**:

1	$(\forall x)(\exists y)[P(x) \rightarrow Q(y)]$	premise
2	$\quad P(a)$	assumption
3	$\quad (\exists y)[P(a) \rightarrow Q(y)]$	$\forall\mathcal{E}$, 1
4	$\quad P(a) \rightarrow Q(\hat{b})$	$\exists\mathcal{E}$, 2
5	$\quad Q(\hat{b})$	Modus Ponens, 2, 4
6	$Q(\hat{b})$	copy **mistake**: out of bounds, 5
7	$(\exists y)Q(y)$	$\exists\mathcal{I}$, 6

The mistake is that $Q(\hat{b})$ left the enclosed scope of the assumption $P(a)$.

5. **Implicit Binding**:

1	$(\forall x)(\exists y)[P(x) \rightarrow Q(y)]$	premise
2	$(\exists y)[P(a) \rightarrow Q(y)]$	$\forall\mathcal{E}$, 1
3	$\quad P(a) \rightarrow Q(\hat{b})$	assumption
4	$\quad (\forall x)[P(x) \rightarrow Q(\hat{b})]$	$\forall\mathcal{I}$ **mistake**: a is under assumption, 3
5	$\quad (\exists y)(\forall x)[P(x) \rightarrow Q(y)]$	$\exists\mathcal{I}$, 4
6	$(\exists y)(\forall x)[P(x) \rightarrow Q(y)]$	$\exists\mathcal{E}$, 2

This binding of a to $\hat{b}$ is only *implicitly* apparent in the *flat* interpretation of $\exists\mathcal{E}$.

1	$(\forall x)(\exists y)[P(x) \rightarrow Q(y)]$	premise
2	$(\exists y)[P(a) \rightarrow Q(y)]$	$\forall\mathcal{E}$, 1
3	$P(a) \rightarrow Q(\hat{b})$	$\exists\mathcal{E}$, 2
4	$(\forall x)[P(x) \rightarrow Q(\hat{b})]$	$\forall\mathcal{I}$ **mistake**: a is *bound* to $\hat{b}$, 3
5	$(\exists y)(\forall x)[P(x) \rightarrow Q(y)]$	$\exists\mathcal{I}$, 4

Contrast these with the following derivation:

1	$(\forall x)(\forall y)[P(x) \rightarrow Q(y)]$	premise
2	$(\forall y)[P(a) \rightarrow Q(y)]$	$\forall\mathcal{E}$, 1
3	$P(a) \rightarrow Q(b)$	$\forall\mathcal{E}$, 2
4	$(\forall x)[P(x) \rightarrow Q(b)]$	$\forall\mathcal{I}$, 3
5	$(\forall y)(\forall x)[P(x) \rightarrow Q(y)]$	$\forall\mathcal{I}$, 4

Here a was not involved in any unfinished assumption when $\forall\mathcal{I}$ was applied on line 4.