CS81 **Assignment 4**

Due Wednesday, 20 February 2013

1. [20 points] Prove the following by natural deduction. Use lemmas when they help make your proof clearer.

   $(\forall x A(x)) \rightarrow \exists x B(x) \vdash \exists x (A(x) \rightarrow B(x))$

For each of the entailments in 2-3 below, use **interpretations** (rather than a natural deduction proof) to argue that the entailment is true, or to give a simple counterexample interpretation if it is false:

2. [10 points] $\forall x(A(x) \vee B(x)) \models (\forall x A(x)) \vee (\exists x B(x))$

3. [10 points] $\forall x(A(x) \vee B(x)) \models (\forall x A(x)) \vee (\forall x B(x))$

4. [10 points] Prove the following using natural deduction with equality rules:

   $\forall x \forall y \forall z\ ((P(x, z) \wedge P(y, z)) \rightarrow x = y)$
   $\vdash (P(a, c) \wedge P(b, d) \wedge P(c, e) \wedge P(d, e)) \rightarrow a = b$

   (It is possible for more than one constant symbol to be interpreted as the same object.)

5. [10 points] Using this set of axioms for Groups:

   A1: $\forall x\ \forall y\ \forall z\ ((x + y) + z = x + (y + z))$
   A2: $\forall x\ ((x + 0 = x) \wedge (0 + x = x))$
   A3: $\forall x\ ((x + (-x) = 0) \wedge ((-x) + x = 0))$

   along with equality rules, prove the following:

   $\forall x \forall y\ -(x+y) = (-y)+(-x)$

6. [20 points]
Below are axioms for the Natural Numbers:
Here ’ is a 1-ary function symbol denoting successor, written in postfix form, + is a 2-ary function symbol, 0 is a constant symbol, and = is equality.

N1: $\forall x\ \forall y\ (x' = y' \rightarrow x = y)$
N2: $\forall x\ \neg(x' = 0)$
N3: $\forall x\ (x + 0 = x)$
N4: $\forall x\ \forall y\ (x + y' = (x + y)')$
Induction: For any formula φ, and variable z:

$$(\varphi[0/z] \wedge \forall z\ (\varphi \rightarrow \varphi[z' / z])) \rightarrow \forall z\ \varphi$$

Note that Induction is an infinite set of axioms, one for each formula φ. We also have the equality rules for +, ’, and =.

Axioms N3 and N4 essentially define addition by induction.

Using natural deduction, prove from the above axioms the associative law of addition:

$$\forall x\ \forall y\ \forall z\ [(x + y) + z = x + (y + z)]$$

This is probably best done by using ∀-Introduction for x and y, which would give an inner statement (where u and v are fresh variables):

$$\forall z\ [(u + v) + z = u + (v + z)]$$

Then prove that statement using induction.

7. [20 points] Let S(x) mean that x is a set. Let C(x, y) mean that x contains y. State as formulas:
   a. There is a set.
   b. There is a set that contains all and only sets that do not contain themselves.

   Prove, using natural deduction: a $\vdash \neg$b.