All proofs are informal and intended to develop intuition and expository style, so please provide convincing write-ups of each.

1. **[5 points]** What is the smallest alphabet $\Sigma$ such that all Turing machines can be encoded as members of $\Sigma^*$?

2. **[20 points]** Consider the language of Turing machine descriptions $L = \{<M> \mid M \text{ accepts at least 999 strings}\}$. Is $L$ decidable, recognizable, or neither? Prove your answer.

3. **[15 points]** Consider the language of Turing machine with tape descriptions $L = \{<M, w> \mid M \text{ accepts } w \text{ using at most 999 tape cells}\}$. Show that $L$ is decidable. (Hint: For how many steps can $M$ run without either accepting, rejecting, using more than 999 cells or going into an infinite loop?)

4. **[20 points]** Consider the language of Turing machine descriptions $L = \{<M> \mid M \text{ accepts all strings of length 999 or longer}\}$. Is $L$ decidable, recognizable, or neither? Prove your answer.

5. **[15 points]** Show that the special case of PCP (Post’s Correspondence Problem) over a 1-letter alphabet is decidable.

**Problems involving mapping reducibility $\leq_m$**

6. **[10 points]** Suppose $R \subseteq \Sigma^*$ is a regular language other than $\emptyset$ and $\Sigma^*$. Let $L \subseteq \Sigma^*$ be any decidable language. Show that $L \leq_m R$ (L is mapping-reducible to R).

7. **[5 points]** Regarding the previous problem, is the same reduction true if L is recognizable but not decidable? Why or why not?

8. **[10 points]** Show that a language $L$ is recognizable iff $L \leq_m A_{TM}$ (the acceptance language for Turing machines).