Predicate Logic

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February 2013

Examples of Predicate Logic Formulas

- $p(x)$
- $q(a)$
- $r(x, f(a))$
- $p(x) \rightarrow r(x, f(a))$
- $q(a) \rightarrow \exists x\ r(x, f(a))$
- $\forall y\ (q(y) \rightarrow \exists x\ r(x, f(y))$
- $\exists x\ q(z) \wedge \forall y\ (q(y) \rightarrow \exists x\ r(x, f(y))$

Predicate Calculus Abstract Syntax

- E is the start symbol
- E ::= A | // Atom or atomic formula
  ¬E | // Negation (not)
  E ∧ E | // Conjunction (and)
  E ∨ E | // Disjunction (or)
  E → E | // Implication (implies)
  E ↔ E | // If-and-only-if
  ⊥ | // Bottom
  T | // Top
  ∀V E | // Universally-quantified formula
  ∃V E // Existentially-quantified formula
- V means **variable symbol** (see next page)
- Precedence, tightest first: ∀ ∃ ¬ ∧ ∨ → ↔
- Atomic formula (A) requires a more complex production

Atomic Formulas

- Informally, an atomic formula is the smallest unit that evaluates to a truth value {T or F}, once individuals are substituted for arguments.
- Atomic formulas don't contain connectives or quantifiers.
- They are analogous to proposition symbols in proposition logic.

Examples of Predicate Logic Formulas

Atomic Formulas Circled

- $p(x)$
- $q(a)$
- $r(x, f(a))$
- $p(x) \rightarrow r(x, f(a))$
- $q(a) \rightarrow \exists x\, r(x, f(a))$
- $\forall y\, (q(y) \rightarrow \exists x\, r(x, f(y)))$
- $\exists x\, q(z) \wedge \forall y\, (q(y) \rightarrow \exists x\, r(x, f(y))$

Atomic Formula Syntax

A ::= P(L)   // Predicate applied to list of terms

L ::= T | T ‘,’ L   // List of terms

T ::= V | C | F(L)   // **Term**

V ::= ‘x’ | ‘y’ | ‘z’ | …   // Variable symbols

P ::= ‘p’ | ‘q’ | ‘r’ | …   // Predicate symbols

C ::= ‘a’ | ‘q’ | ‘c’ | …   // Constant symbols

F ::= ‘f’ | ‘g’ | ‘h’ | …   // Function symbols

Some predicates and functions may be abbreviated in infix form, e.g.

= < < . . . will be infix predicate symbols

+ * / . . . will be infix function symbols

We will not bother with a special grammar for these, although it can be done.

Arities

- In addition, predicate and function symbols have an “arity” (number of arguments) which we don’t show explicitly.

- Most of the time, we will **not overload** the symbols, but rather assume a fixed arity for a given symbol.

- So we will **not** typically use both f(a, b) (2–ary) and f(a) (1–ary), for example, in the same discussion.

Quantifiers

- $\forall$ is a "wholesale" version of $\wedge$
- $\exists$ is a "wholesale" version of $\vee$
- $\forall x\ P(x)$ is like $P(x_0) \wedge P(x_1) \wedge P(x_2) \wedge \ldots$

  except that we don't know how many elements there are.

Quantifiers

- $\forall x\ P(x)$ is like $P(x_0) \wedge P(x_1) \wedge P(x_2) \wedge \ldots$

  just as $\sum_x f(x)$ is like $f(x_0) + f(x_1) + f(x_2) + \ldots$

First-Order Logic

- Our focus here is *first-order logic* (FOL) or *first-order predicate calculus* (FOPC).

  http://en.wikipedia.org/wiki/First-order_logic

- Second-, and higher-, order logic would include quantification over predicates and functions. It is not within the scope of this course, but may get brief mention.

  http://en.wikipedia.org/wiki/Higher-order_logic

“Term”: a term to remember

- A ***term*** designates an individual in a **domain** (to be introduced later).

- A term can be:
  - A **constant symbol**, naming the individual
  - A **variable symbol**, naming a generic individual
  - A **function** applied to some terms as arguments, the result of which is **the individual the function produces**.

Examples of Terms

• b	constant symbol
• y	variable
• f(b, y)	function applications
• g(h(b), c, h(y))	
• g(a, b, g(a, b, c))	

- Atomic formulas are not terms, although they look similar.
- Atomic formulas can *contain* terms as arguments.

Examples of Atomic Formulas

- p(b)
- q(y)
- p(f(b, y))
- r(a, g(h(b), c, h(y)))

The arguments must be terms.

Examples of “Literals”

- A **literal** is an atomic formula, or the **negation** of an atomic formula.

  p(b)
  ¬q(y)
  ¬p(f(b, y))
  r(a, g(h(b), c, h(y)))

- Literals become important in resolution theorem proving, discussed later.

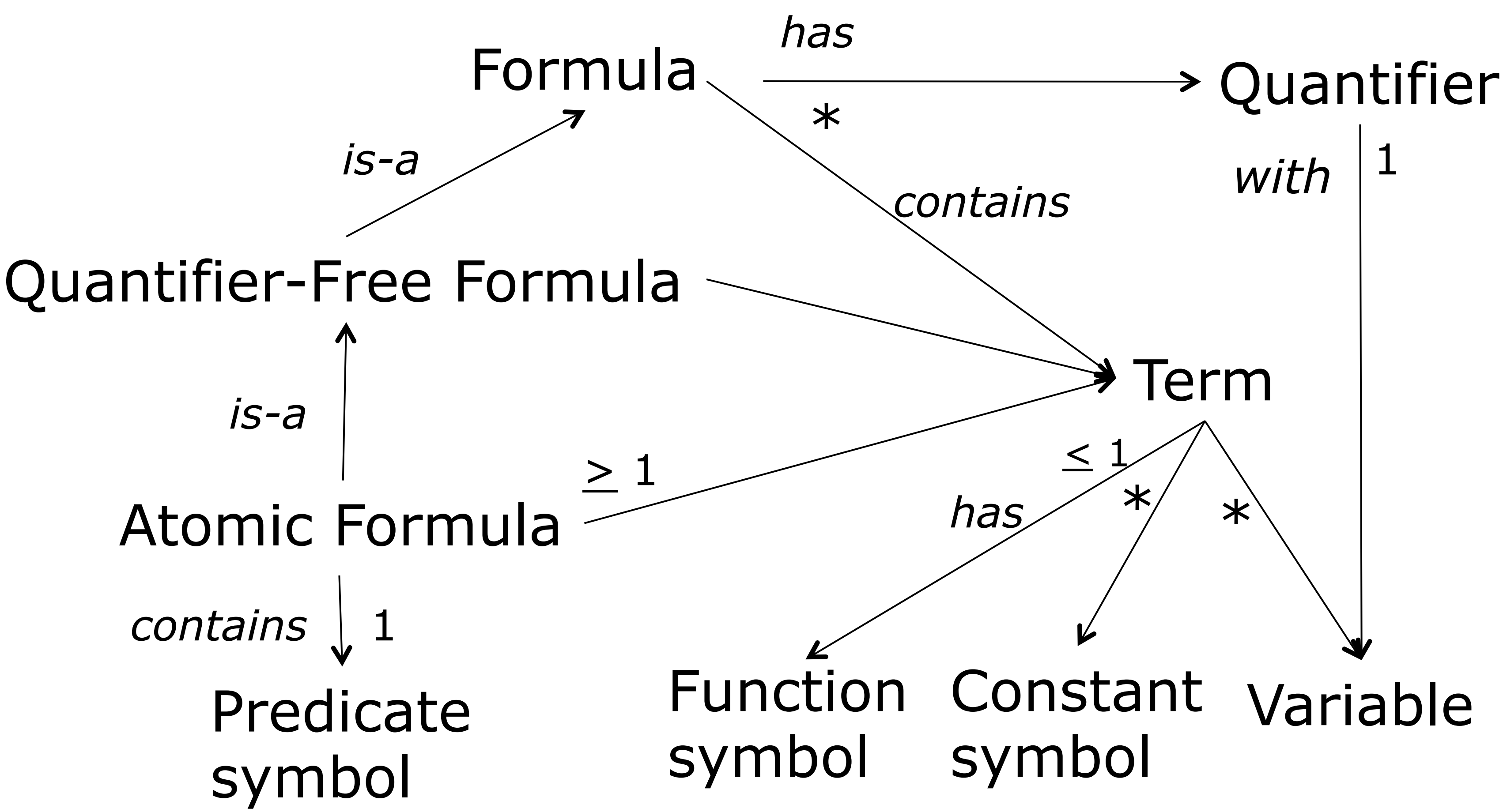
Examples of **Quantifier-Free** Formulas

- Any atomic formula
- $p(b) \vee p(c)$
- $p(y) \wedge q(y)$
- $p(f(b, y)) \rightarrow q(y)$
- $\neg r(a, g(h(b), c, h(y)))$

Examples of Formulas

- Any Quantifier-Free Formula
- $\exists x\ p(x)$
- $\forall y\ (p(y) \wedge q(y))$
- $\forall y\ \exists x\ (p(f(x, y)) \rightarrow q(y))$
- $\forall x\ (p(f(x, y)) \vee q(x))$
- $\forall y\ (q(y) \rightarrow \exists x\ p(f(x, y)))$

Structural Summary



* means “zero or more”

“Well-Formed Formulas”

- WFFs (sometimes pronounced “woofs”)
- We don’t deal with formulas that are *not* well-formed. In the second part of the course, we discuss grammars and parsing.
- See http://en.wikipedia.org/wiki/Well-formed_formula

Preview of Semantics

- We will give details of semantics later on. However, a preview is helpful to understand certain syntactic considerations.
- Predicate logic can be used to describe characteristics of particular kinds of **structures**, such as sets with certain algebraic properties or real-world objects.
- The particular structures are called "Interpretations". Interpretations which make a set of formulas true are called "models".

Example:

Interpretation for the natural numbers

- The intended domain is {0, 1, 2, 3, …}.
- There is a constant symbol **0**.
- There is a 1-ary function **s** (successor).

  Informally, s(n) = n+1.
- There is a 2-ary predicate **=** (equals).

Individuals in the Domain

- Any individual natural number can be described by a term

- 0 describes the number ‘0’
- s(*t*) where *t* is a term, describes 1+ the number described by *t*.

- s(0) describes ‘1’
- s(s(s(s(s(s(s(s(s(s(0)))))))))) describes ____?

Some formulas for this interpretation

- $\forall n \neg(s(n) = 0)$

  "0 is not the successor of anything".

- $\forall m (\neg(m = 0) \rightarrow (\exists n) (m = s(n)))$

  "Anything other than 0 is the successor of something".

- $\forall m \forall n ((s(m) = s(n)) \rightarrow m = n)$

  "Successor is a one-to-one function".

Example:
Interpretations for “Groups”

- The domain is non-empty.
- The domain can be finite or infinite.
- There is a constant symbol **e** (**identity element** of the group).
- There is a 2-ary function **f** (group “multiplication”).
- There is a 2-ary predicate **=** (equals).

Some formulas for groups

- $\forall x \; f(e, x) = x$

  [e is an identity]

- $\forall x \, \forall y \, \forall z \; f(x, f(y, z)) = f(f(x, y), z)$

  [f is associative]

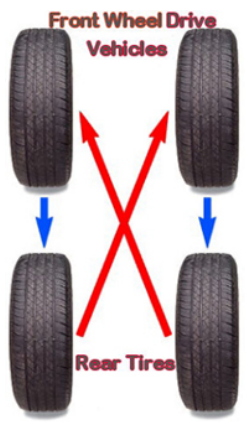
- $\forall x \, \exists y \; f(x, y) = e$

  [existence of inverse]

Examples of Groups

- Trivial group: {0} e = 0, f(0, 0) = 0
- 2-element group: {0, 1} e = 0, f(x, y) = x + y (mod 2)
- Z_p: {0, 1, …, p-1} for any prime p,
  e = 0, f(x, y) = x + y (mod p)
- Tire rotations
- Particle spins (physics)
- Rubik's cube states
- Many others

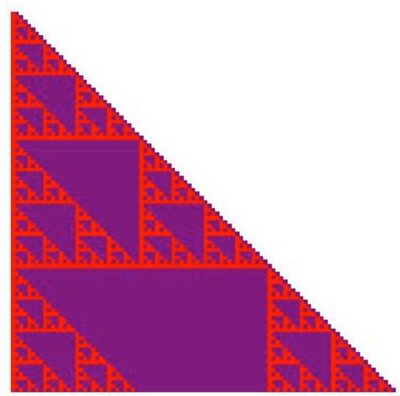
Examples of Groups



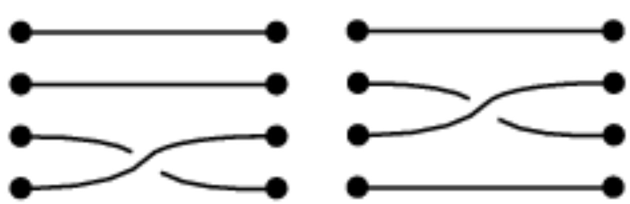
4 elements

$$|\psi\rangle \to e^{-\frac{i}{\hbar}(2\pi)S_z}|\psi\rangle = -|\psi\rangle$$

spins



cellular automaton
based on a group

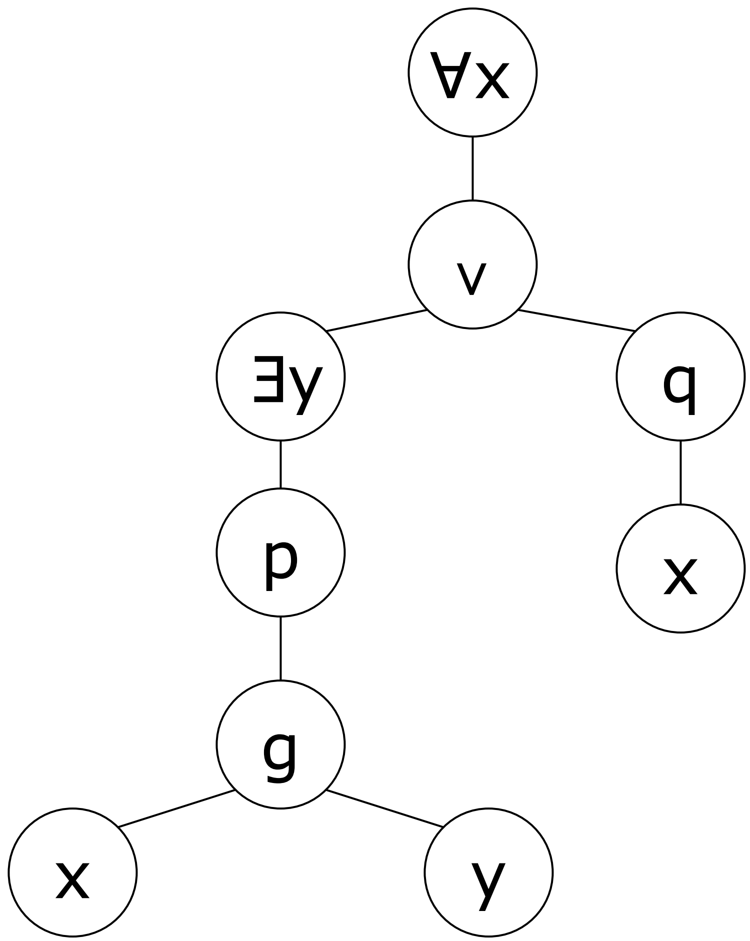


braids

states: 43252003274489856000 elements

Syntax Trees (or “Parse” Trees)

- We are assuming familiarity with syntax trees from CS 60.
- ∀x, ∃x are treated as if **1-ary operators**.
- Example: ∀x ((∃y p(g(x, y))) ∨ q(x))

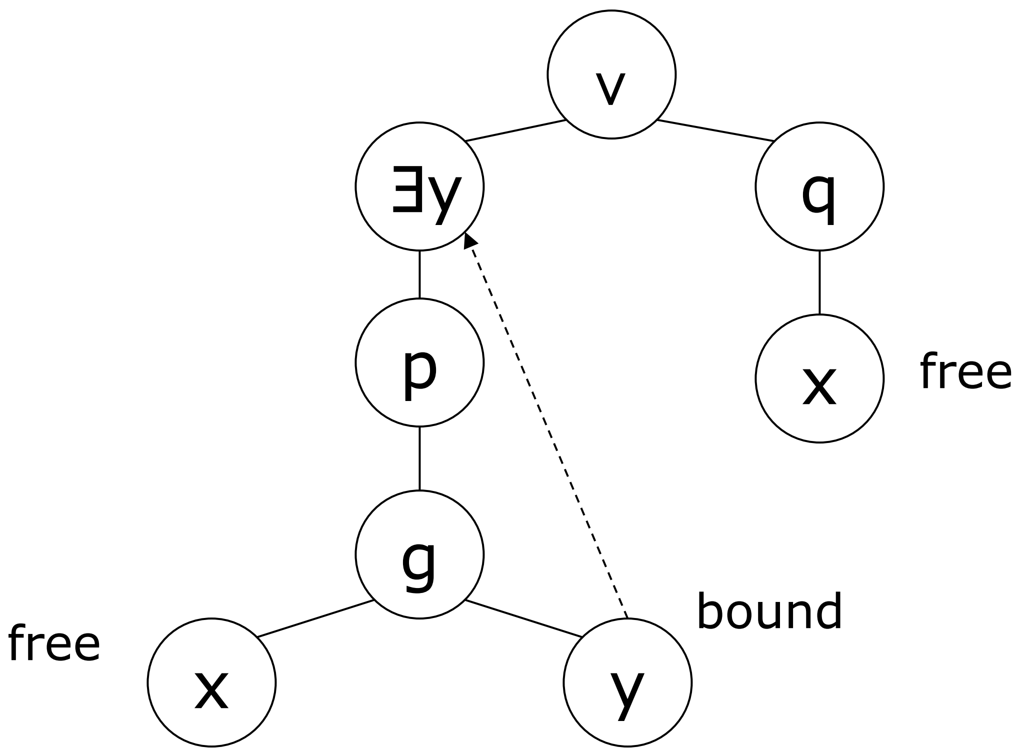


Bound and Free Variables

- A variable *v* is **bound** in a formula if it occurs in a sub-tree having ∃*v* or ∀*v* at the root.

- Otherwise it is **free** in the formula.

Free and Bound Variable Instances



Analogues in Calculus

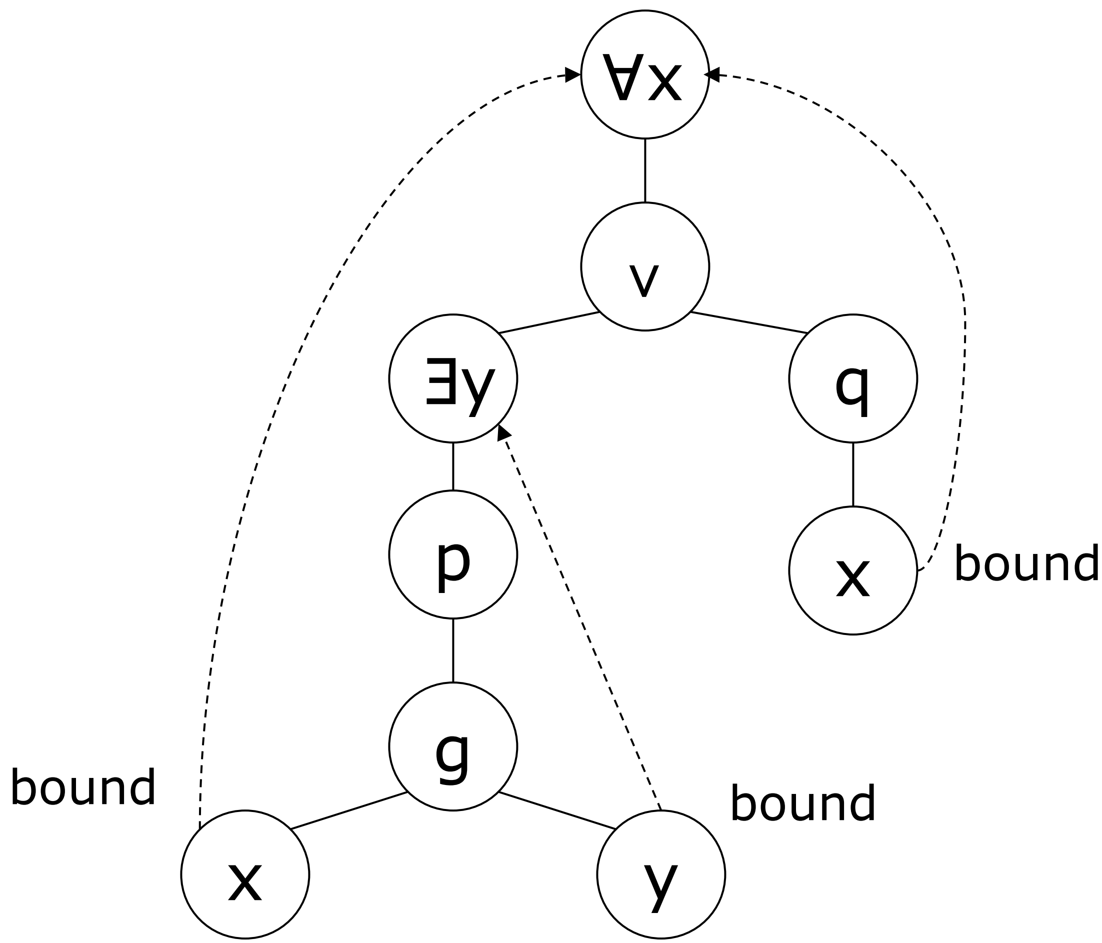
$$\sum_{k=1}^{10} f(k,n)$$

k is bound, n is free

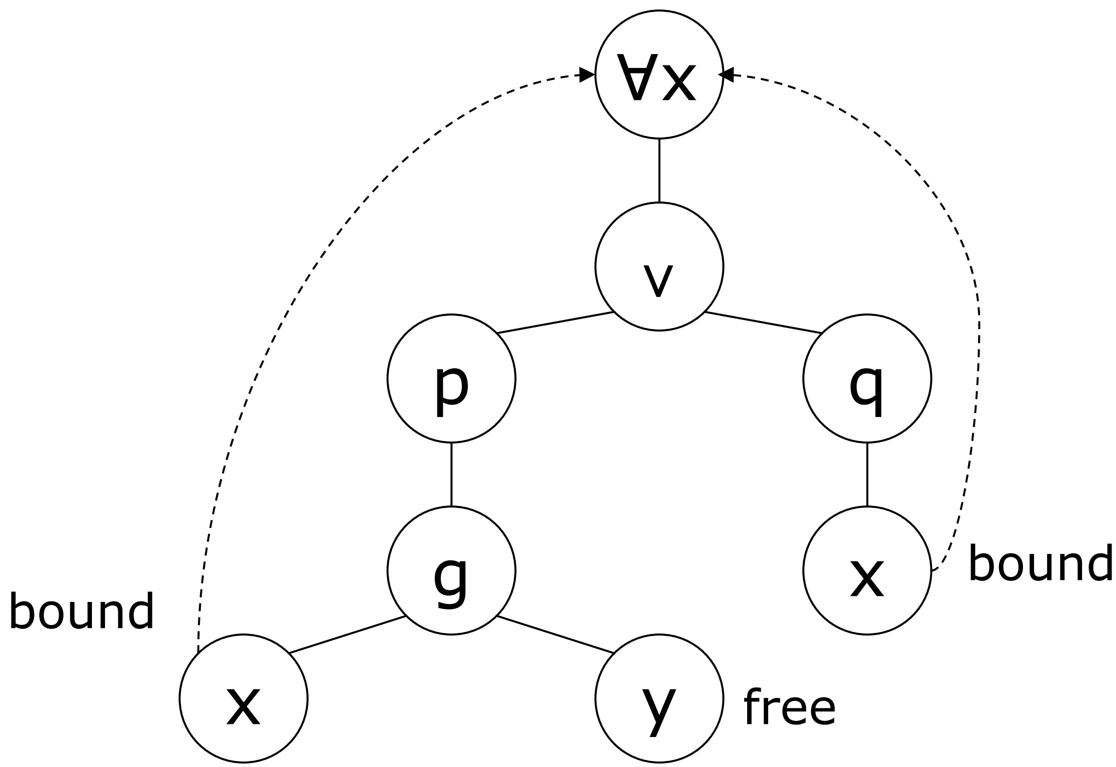
$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

h is bound, x is free

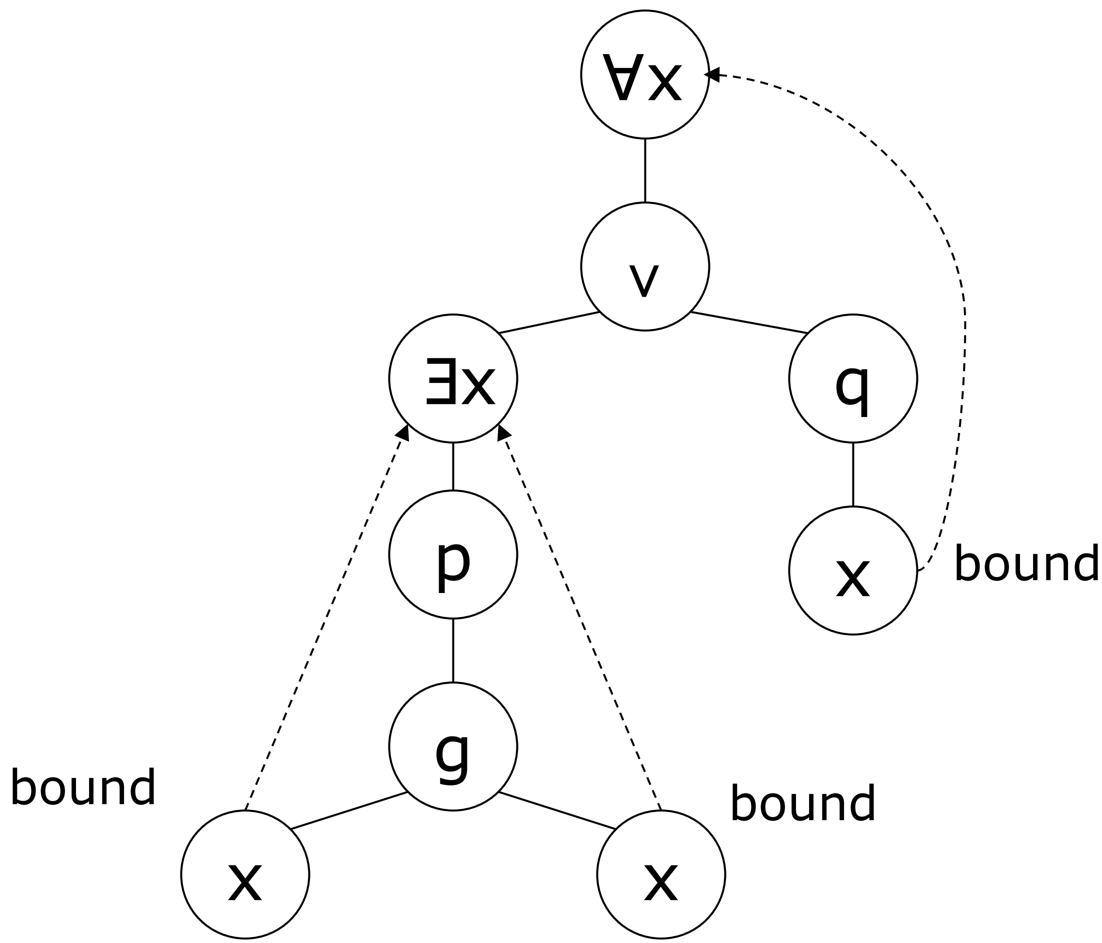
Free and Bound Variable Instances



Free and Bound Variable Instances



Free and Bound Variable Instances



Scope of Variables

- The same variable may be used more than once in a formula, with different “meanings”.
- The idea of **scope** clarifies these separate meanings.
- For a formula $\forall x$ E, or $\exists x$ E, the scope of x extends only inside E, and not beyond.
- Similar to scope in programming languages

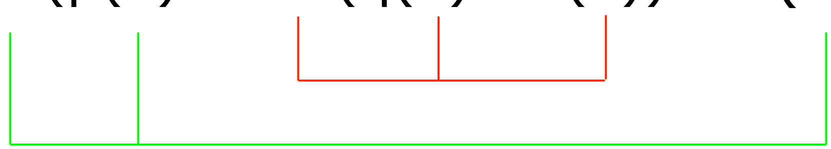
Scope Defined Inductively

- For a quantifier-free formula, the scope of each variable is the entire formula.

- For $\forall x$ E, or $\exists x$ E, the scope of x is inside E, but not including inside any quantification of the same variable inside E.

- Example: Two distinct scopes of variable x:

$$\forall x\ (p(x) \vee \exists x\ (q(x) \wedge r(x)) \vee s(x, y))$$



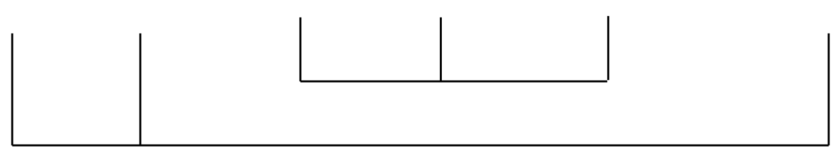
Renaming Variables

- Although not required, it is better to avoid using the same variable for more than one scope.

- Bound variables can be **renamed to a fresh variable** to accomplish this. All instances within the scope must be renamed.

- Example: One of the x's renamed to u:

$$\forall x\ (p(x) \lor \exists u\ (q(u) \land r(u)) \lor s(x, y))$$



Improper renaming

- $\forall x\ (p(x) \vee \exists x\ (q(x) \wedge r(x)) \vee s(x, y))$
- Can’t rename **just one** of the inner x’s
- $\forall x\ (p(x) \vee \exists u\ (q(u) \wedge r(x)) \vee s(x, y))$
- The scope of the x in r(x) would change.

Definition of Free and Bound Instances

- In a **term** and in a **quantifier-free formula**, every instance of a variable is free.

- If φ is a formula, then any **free instances** of a variable x **are bound** in $\forall x\ \varphi$ and $\exists x\ \varphi$.

- The free instances of variables in φ and ψ **remain free** in $(\neg\varphi)$, $(\varphi\vee\psi)$, $(\varphi\wedge\psi)$, and $(\varphi \rightarrow \psi)$.

- The bound instances of variables in φ and ψ **remain bound** in $(\neg\varphi)$, $(\varphi\vee\psi)$, $(\varphi\wedge\psi)$, and $(\varphi \rightarrow \psi)$.

Substitutability Restriction

- We are going to need to be able to **substitute terms** for ***free* variables** in various formulas.

- While this is easy syntactically, there is a semantic restriction that must be observed:

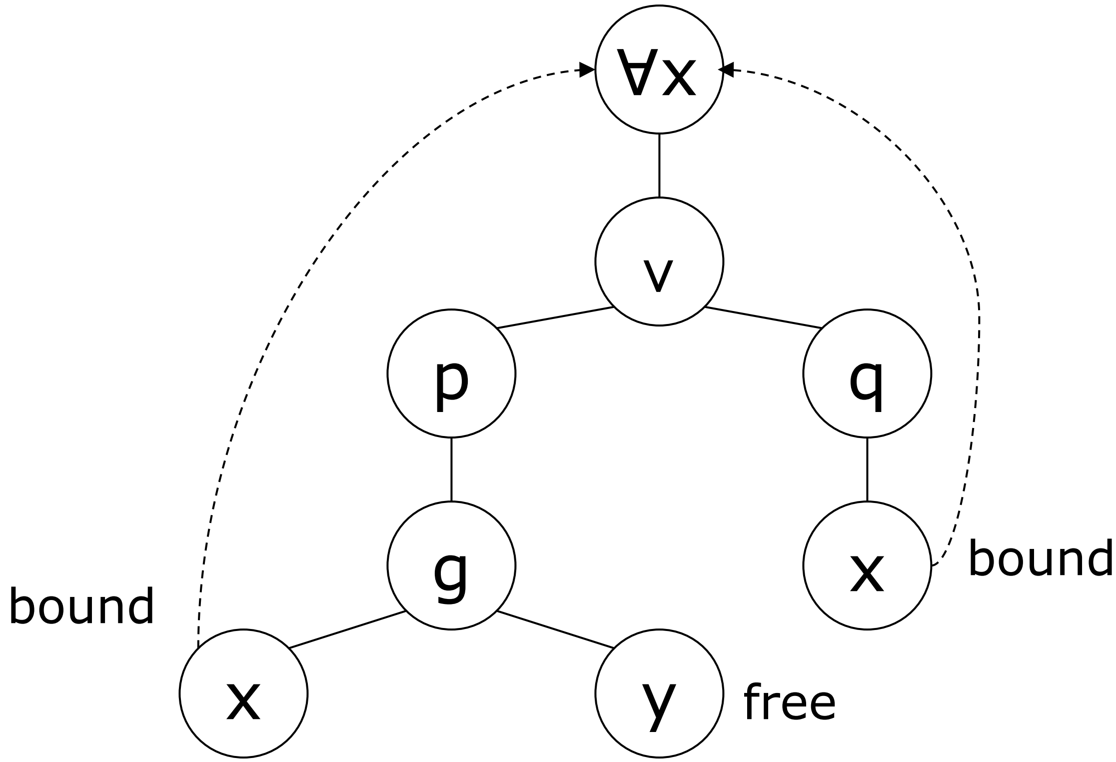
  - **In substituting a term** for a variable within a formula, **no variables *within* the term can become bound** as a result of the substitution.

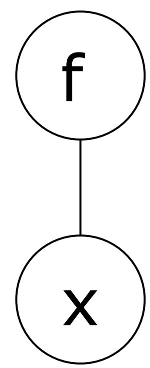
- If t is a term, v is a variable, and F is a formula, and the above restriction applies, we say that

  "t is free to replace v in F"

  (or more conventionally, **"t is free for v in F")**

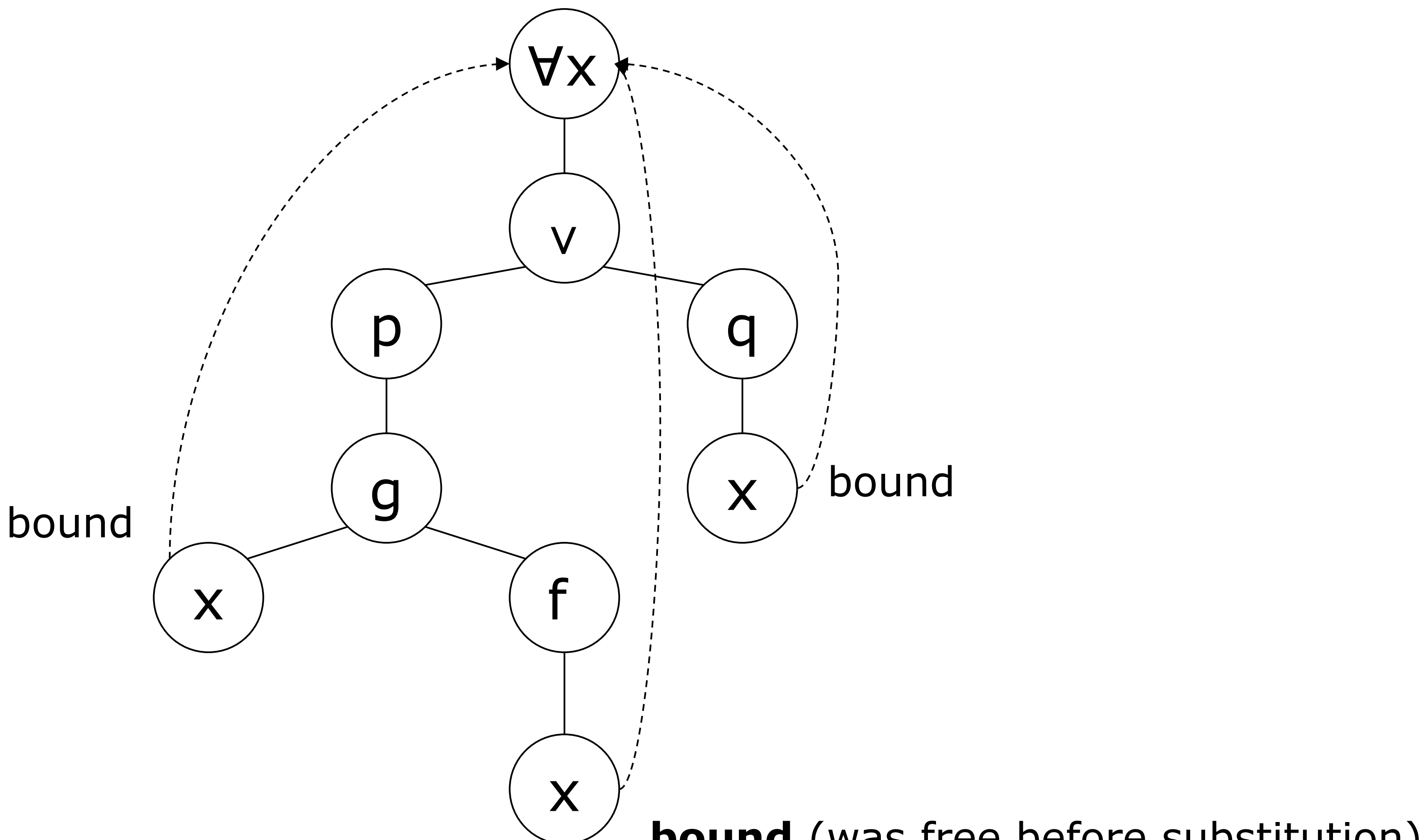
Non-Substitutability Example





f(x) is **not** free to replace y in
∀x (p(g(x, y)) ∨ q(x))

Non-Substitutability Example



∀x (p(g(x, **f(x)**)) v q(x))

Example of Violation

- $\exists x\ x<y$
- We don't allow substitution of x for y because the meaning would change:
- $\exists x\ x<x$

Substitution Notation

- If t is a term, v is a variable, and E is a formula, and

  t is free to replace v in E

  then by

  E[t/v]

  we mean the result of substituting t for every **free** occurrence of v in E. (We leave the bound occurrences of v as they were.)

  This notation and substitution itself are to be used **only** when the **substitutability restriction applies**.

  Note: [/] is **meta**-syntax; these symbols do not appear in the resulting formula.

Substitution Notation Example

Let E be the formula

$\forall x \, (p(g(x, y)) \vee q(x))$

Let v be the variable y.

Let t be the term f(z).

f(z) **is** free to replace y in E.

E[f(z)/y] is $\forall x \, (p(g(x, f(z))) \vee q(x))$.

Substitution Notation Example

Let E be the formula

$\forall x\ (p(g(x, y)) \vee q(x))$

Let v be the variable x.

Let t be the term f(y).

f(x) **is** free to replace x in E (vacuously)
because there are **no free instances** of x.

E[f(x)/x] **is the same as** E;
there are no free instances of x in E.

A simpler way of writing substitutions

If E is a formula, then E(x)
identifies any *free* occurrences of x in E.
(There might not be any.)

If t is a term free for x in E, then E(t)
is the result of substituting t for all free instances of x.

Example

- Suppose E is $\forall x\ (p(g(x, y)) \vee q(y))$

- Then E(y) has y identified with the two occurrences of y.

- f(z) is free for y in E.

- E(f(z)) is $\forall x\ (p(g(x, f(z))) \vee q(f(z)))$

Syntax vs. Semantics

- Predicate logic proofs, in a system such as natural deduction, focus on **syntax**: each formula in the derivation is **mechanically-checkable** to be derivable from earlier formulas using only the given rules.

- The **semantics** or **meaning** of a formula is determined by separate considerations.
  Each formula is making a statement about some kind of **underlying structure**.

Why Separate Syntax from Semantics?

- Reasoning about semantics is often very complex.

- Reasoning syntactically allows reasoning without revisiting semantic details at every step.

Natural Deduction Rules for Predicate Logic

Natural Deduction Rules

- We need introduction and elimination rules for both:
  - $\forall$
  - $\exists$
- These will be added to our propositional natural deduction rules.

∀-Elimination Rule ∀E

- $$\frac{\forall x\ \varphi}{\varphi[t/x]} \quad \forall E$$

  where t is any term that is free to replace x in φ.

- **What the rule says:**

  If we have derived a universally-quantified formula φ,
  then the formula φ with any (appropriately-qualified) **specific instance** of x substituted for x is also derivable.

Two ways of writing

- $$\frac{\forall x\ \varphi}{\varphi[t/x]}\ \forall E$$

- $$\frac{\forall x\ \varphi(x)}{\varphi(t)}\ \forall E$$

Why the Substitution Qualification is Necessary

- $\dfrac{\forall x\ \varphi}{\varphi[t/x]}$ ∀E_

  where t is any term that is free to replace x in φ.

- Correct example: z is free to replace x in ∃y p(y, x)
  1. ∀x ∃y p(y, x) Premise
  2. ∃y p(y, z) ∀E 1 (substituting **z** for x)

- Incorrect example: y is **not** free to replace x in ∃y p(y, x)
  1. ∀x ∃y p(y, x) Premise
  2. ∃y p(y, y) ∀E 1 (substituting **y** for x)

- For instance, p could be > in the domain of natural numbers.

∀-Introduction Rule (∀I)

- This rule uses a sub-derivation, with **no formula assumed**, but with a **fresh variable** introduced.

Fresh x_0
.
.
.
$\varphi[x_0/x]$

$$\frac{\boxed{\text{Fresh } x_0 \;\; \vdots \;\; \varphi[x_0/x]}}{\forall x\ \varphi}\ \forall I$$

- x_0 is a "fresh" variable otherwise unused in the proof.
- x_0 must be free to replace x in φ, but since x_0 is "fresh", this should never be an issue. It can't become bound.

Another way of writing

Fresh x_0

.

.

.

$\varphi(x_0)$

$$\frac{\boxed{\text{Fresh } x_0 \; \vdots \; \varphi(x_0)}}{\forall x\, \varphi} \; \forall I$$

∀-Introduction Rule

- **What this rule says:**
- If we have argued to derive a term $\varphi[x_0/x]$ where x_0 represents a **totally arbitrary** value of x, then we are justified in concluding $\forall x\ \varphi$.
- The key is the word "arbitrary"; there can be **no constraints** attached to x_0.
- Note: Once the conclusion $\forall x\ \varphi$ is drawn, x_0 is **discharged** and cannot be further used outside the box.

Use ∀I working backward

Unless your **goal** is proving something of the form ∀x … you won't know to open a box with a fresh variable.

∀E ∀I Example

Derive $\forall x\ p(x) \vdash \forall y\ p(y)$

$\forall x\ p(x)$	Premise
$\forall y\ \ p(y)$	Desired conclusion

Work backward

∀E ∀I Example

Derive $\forall x\ p(x) \vdash \forall y\ p(y)$

1.		$\forall x\ p(x)$	Premise
2.	y_0		Fresh var
3.		$p(y_0)$	1, ∀E
4.	$\forall y\ \ p(y)$		2-3, ∀I

∀E ∀I Example

Derive $\forall x\,(p(x) \rightarrow q(x)), \forall x\, p(x) \vdash \forall x\, q(x)$

1.	$\forall x\,(p(x) \rightarrow q(x))$	Premise
2.	$\forall x\ \ p(x)$	Premise
3.	x_0	Fresh var
4.		
5.	$q(x_0)$	
6.	$\forall x\, q(x)$	3-6, ∀I

∀E ∀I Example

Derive $\forall x\, (p(x) \rightarrow q(x)),\ \forall x\, p(x) \vdash \forall x\, q(x)$:

1.	$\forall x\, (p(x) \rightarrow q(x))$	Premise
2.	$\forall x\ \ p(x)$	Premise
3.	x_0	Fresh var
4.	$p(x_0) \rightarrow q(x_0)$	1, ∀E
5.	$p(x_0)$	2, ∀E
6.	$q(x_0)$	4, 5 →E
7.	$\forall x\ q(x)$	3-6, ∀I

$\forall$E $\forall$I English Equivalent

- Derive $\forall x\,(p(x) \rightarrow q(x)), \forall x\, p(x) \vdash \forall x\, q(x)$:
- "Assume $\forall x\,(p(x) \rightarrow q(x))$ and $\forall x\, p(x)$.

  Let x_0 be an arbitrary element. [open box]

  From the the first assumption $p(x_0) \rightarrow q(x_0)$, and from the second $p(x_0)$, hence also $q(x_0)$ by *modus ponens*.

  Since x_0 was chosen arbitrarily, $q(x_0)$ gives us [close box] $\forall x\, q(x)$."

∀E ∀I Example

Derive $\forall x\ \forall y\ p(x, y) \vdash \forall y\ \forall x\ p(x, y)$

$\forall x\ \forall y\ p(x, y)$ Premise

$\forall y\ \forall x\ p(x, y)$

Where ∀I is to be used, work backward.

∀E ∀I Example

Derive $\forall x\ \forall y\ p(x, y) \vdash \forall y\ \forall x\ p(x, y)$

1.		$\forall x\ \forall y\ p(x, y)$	Premise
2.	y_0		Fresh
3.	x_0		Fresh
4.		$\forall y\ p(x_0, y)$	1, ∀E
5.		$p(x_0, y_0)$	4, ∀E
6.		$\forall x\ p(x, y_0)$	3-5, ∀I
7.		$\forall y\ \forall x\ p(x, y)$	2-6, ∀I

∃-Introduction Rule (∃I)

- $\dfrac{\varphi[t/x]}{\exists x\ \varphi}$ (∃I)

  where t is any term that is free to replace x in φ.

- **What the rule says:**

  If we have exhibited a formula φ in which variable x is replaced by a **specific instance** t (a term) **then** we can conclude that there is **an** x for which the formula is true.

∃-Introduction Rule (∃I)

- $$\frac{\varphi[t/x]}{\exists x\ \varphi}\ \exists I$$

  where t is any term that is free to replace x in φ.

- In essence, this rule **loses information**, by replacing knowledge of a **specific** x for which is true with the statement that there is some such x.

- It is analogous to rule $\vee$-Introduction.

Another way of writing

- $\underline{\varphi(t)}$ ∃I
  $\exists x\ \varphi(x)$

Why would you want to lose information?

- For one thing, the specific term t derived might not be “exportable”;

  it could depend on some fresh variable introduced inside the box, which doesn’t make sense outside.

∀E ∃I Example

- Assume there is a constant symbol a.
- Derive ∀x p(x) $\vdash$ ∃x p(x) :

1.	∀x p(x)	Premise
2.	p(a)	1, ∀E
3.	∃x p(x)	2, ∃I

The previous example is rare.

- As with ∨ Introduction,

  ∃ Introduction is almost never the last line of a proof when the premise and conclusion are **equivalent**.

Slight Controversy

- What if there are no constant symbols? Use a variable instead.
- Derive $\forall x\ p(x) \vdash \exists x\ p(x)$:

1.	$\forall x\ p(x)$	Premise
2.	$p(x)$	1, $\forall$E
3.	$\exists x\ p(x)$	2, $\exists$I

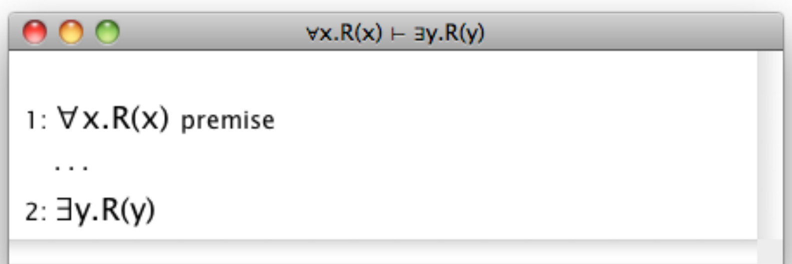
Note: x is free to replace x in p(x),
since nothing is bound in p(x).

The legitimacy of step 2 is questionable. It amounts to **assuming** that there is at least one thing in the domain, i.e. a **non-empty domain**.

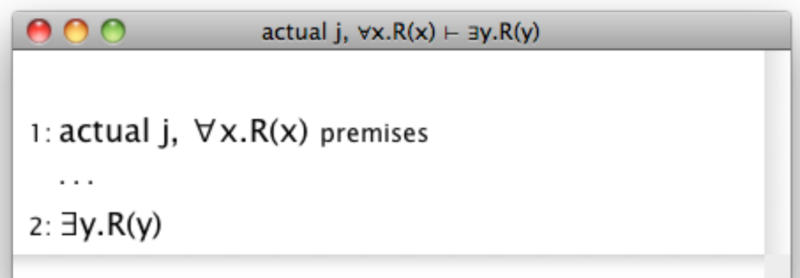
Most treatments assume this, but not all.
For example, Richard Bornat, the author of JAPE does not.
(Allowing empty domains is analogous to allowing 0 as a number.)

JAPE Examples

Not JAPE-Provable (no assumption that something exists)

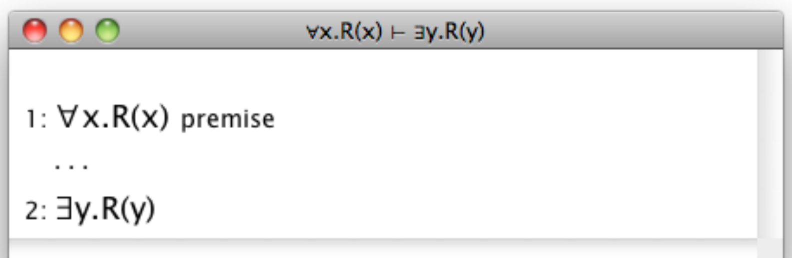


JAPE-Provable (**actual j** means something exists, i.e. a is a constant)

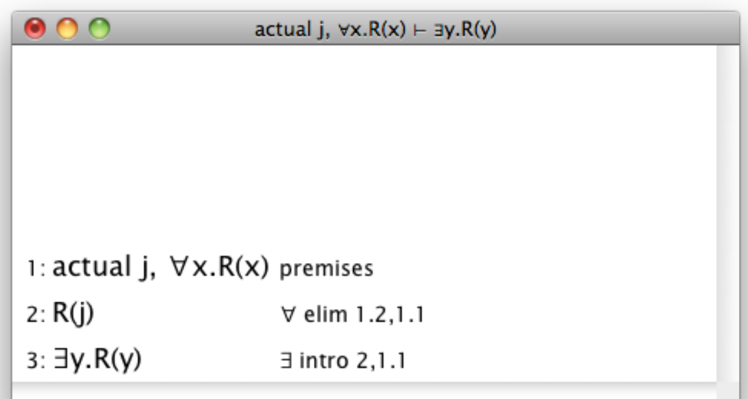


JAPE Examples

This is listed in Invalid Conjectures.



JAPE Proof
with actual j added



Summary

If using JAPE and you want to make the non-empty domain assumption,

include “actual j” as a premise.

Alternatively, if T is available, you could include ∃x.T, from which actual j could be derived. Or you could include $\exists x.\varphi$ where φ is any provable formula.

∃-Elimination Rule (∃E)

$$\dfrac{\exists x \varphi \qquad \boxed{\begin{array}{ll} x_0 & \varphi[x_0/x] \\ & \vdots \\ & \chi \end{array}}}{\chi} \quad (\exists E)$$

- Here x_0 is a “fresh” variable otherwise unused in the proof.
- x_0 must be free to replace x in φ,
  but since x_0 is “fresh”, this should never be an issue.
- This rule is analogous to v Elimination.

∃-Elimination Rule (∃E)

	x_0 $\quad \varphi[x_0/x]$ . . . χ	
$\exists x\ \varphi$		∃E

$$\frac{\exists x\ \varphi \qquad \boxed{\begin{array}{ll} x_0 & \varphi[x_0/x] \\ & \vdots \\ & \chi \end{array}}}{\chi}\ \exists E$$

- **What this rule says:**
- **Assume** that we have derived $\exists x\ \varphi$ (on left). We can make use of this fact by letting x_0 be ***an*** x such that $\varphi[x_0/x]$. There can be no other constraints on x_0. If we then derive χ from the assumption about φ, then we can conclude χ in general.

∃I ∃E Example

- Derive $\forall x\,(p(x) \rightarrow q(x)),\ \exists x\, p(x) \vdash \exists x\, q(x)$:

1.		$\forall x\,(p(x) \rightarrow q(x))$	Premise
2.		$\exists x\, p(x)$	Premise
3.	x_0	$p(x_0)$	Fresh var, Assumption
4.		$p(x_0) \rightarrow q(x_0)$	1, ∀E
5.		$q(x_0)$	3, 4, →e
6.		$(\exists x)\, q(x)$	5, ∃I
7.		$(\exists x)\, q(x)$	2, 3-6, ∃E

always same formula (lines 6 and 7)

- In the ∃E rule template, φ is identified with $p(x)$, while χ is identified with $\exists x\, q(x)$.

- Try not to be confused by the fact that ∃ is in the conclusion. The ∃ in 2 is what was eliminated.

∃E Example is tricky

This rule is the hardest for students to get right, so study the examples carefully.

∃I ∃E Example in English

- Derive $\forall x\,(p(x) \rightarrow q(x)), \exists x\, p(x) \vdash \exists x\, q(x)$:
- “Assume $\forall x\,(p(x) \rightarrow q(x))$ and $\exists x\, p(x)$.

  Let x_0 be such that $p(x_0)$, by the second assumption.

  By the first assumption, $p(x_0) \rightarrow q(x_0)$.
  Hence $q(x_0)$ by modus ponens.

  As we have exhibited an x (namely x_0) such that $q(x)$, conclude $\exists x\, q(x)$.”

∃I ∃E **Incorrect** Proof Example

- Derive $\forall x\ (p(x) \rightarrow q(x)), \exists x\ p(x) \vdash \exists x\ q(x)$:

1.		$\forall x\ (p(x) \rightarrow q(x))$	Premise
2.		$\exists x\ p(x)$	Premise
3.	x_0	$p(x_0)$	∃E
4.		$p(x_0) \rightarrow q(x_0)$	1, ∀E
5.		$q(x_0)$	3, 4, →E
6.		$q(x_0)$	3-5, ∃E
7.		$(\exists x)\ q(x)$	6, ∃I

- Why incorrect?
- Formulas containing x_0 **cannot be carried outside the box**.
- The box for ∃E has **two** purposes:
  - Restricting the scope of the introduced variable x_0.
  - Restricting the scope of the assumption.

Caution: ∃E

- Normally, ∃E can only be used to introduce a variable **once** inside a box. You **cannot** use it to introduce a **second** distinct variable in the same box.
- In other words, $\exists x \varphi$ says that **an** x exists, but **not necessarily more than one** x.
- In contrast, you can use ∃I as many times as you want (not that it will always help).

Quantifier rule summary

	Introduction	**Elimination**
$\forall$	Fresh x_0 . . . $\varphi[x_0/x]$ ——————— $\forall x\ \varphi$ $\forall I$	$\forall x\ \varphi$ ——————— $\varphi[t/x]$ $\forall E$ (t is free to replace x)
$\exists$	$\varphi[t/x]$ ——————— $\exists x \varphi$ $\exists I$ (t is free to replace x)	x_0 $\varphi[x_0/x]$ . . . χ $\exists x\ \varphi$ ——————— χ $\exists E$

Parallels

- ∀ Elimination similar to ∧ Elimination

$$\frac{\forall x\ \varphi}{\varphi[t/x]}\ \forall E$$

(t is free to replace x)

Think of $\forall x\ \varphi(x)$ as $\varphi(x_1) \wedge \varphi(x_2) \wedge \ldots$

$$\frac{F_1 \wedge F_2}{F_1}\ \wedge E \qquad \frac{F_1 \wedge F_2}{F_2}\ \wedge E \quad \ldots \quad \frac{F_1 \wedge F_2 \wedge F_3}{F_3}\ \wedge E$$

Parallels

- ∃ Introduction similar to ∨ Introduction

$$\frac{\varphi[t/x]}{\exists x\ \varphi}\ \exists I$$

(t is free to replace x)

Think of $\exists x\ \varphi(x)$ as $\varphi(x_1) \vee \varphi(x_2) \vee \ldots$

$$\frac{F_1}{F_1 \vee F_2}\ \vee I \qquad \frac{F_2}{F_1 \vee F_2}\ \vee I \quad \ldots \quad \frac{F_3}{F_1 \vee F_2 \vee F_3}\ \vee I$$

Parallels

- ∀ Introduction similar to ∧ Introduction

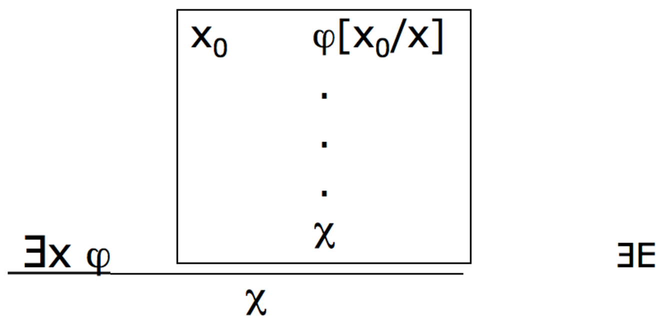
$$\dfrac{\boxed{\begin{array}{l}\text{Fresh } x_0\\ \quad\vdots\\ \varphi[x_0/x]\end{array}}}{\forall x\ \varphi}\ \forall I$$

Think of $\forall x\ \varphi(x)$ as $\varphi(x_0) \wedge \varphi(x_1) \wedge \varphi(x_2) \wedge \ldots$

$$\dfrac{F_1 \qquad F_2 \qquad F_3}{F_1 \wedge F_2 \wedge F_3}\ \wedge I$$

Parallels

- ∃ Elimination similar to ∨ Elimination



Think of $\exists x\ \varphi(x)$ as $\varphi(x_0) \vee \varphi(x_1) \vee \varphi(x_2) \vee \ldots$

In ∨E, there is one box for each disjunct.

More Tips on JAPE Usage

- Introduction rules are both **backward** rules.
- Elimination rules are both **forward** rules.
- ∃I and ∀E don't require special handling.
- ∃E and ∀I require a term to be present for unification:
  - Both require identifying the **term to be substituted** for the quantified variable.

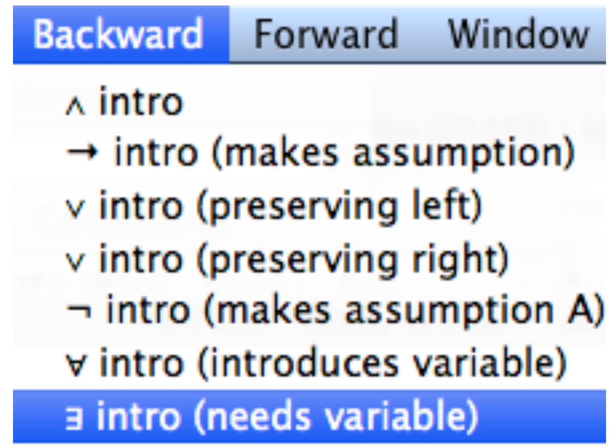
JAPE Example

- Cannot do this, for a reason

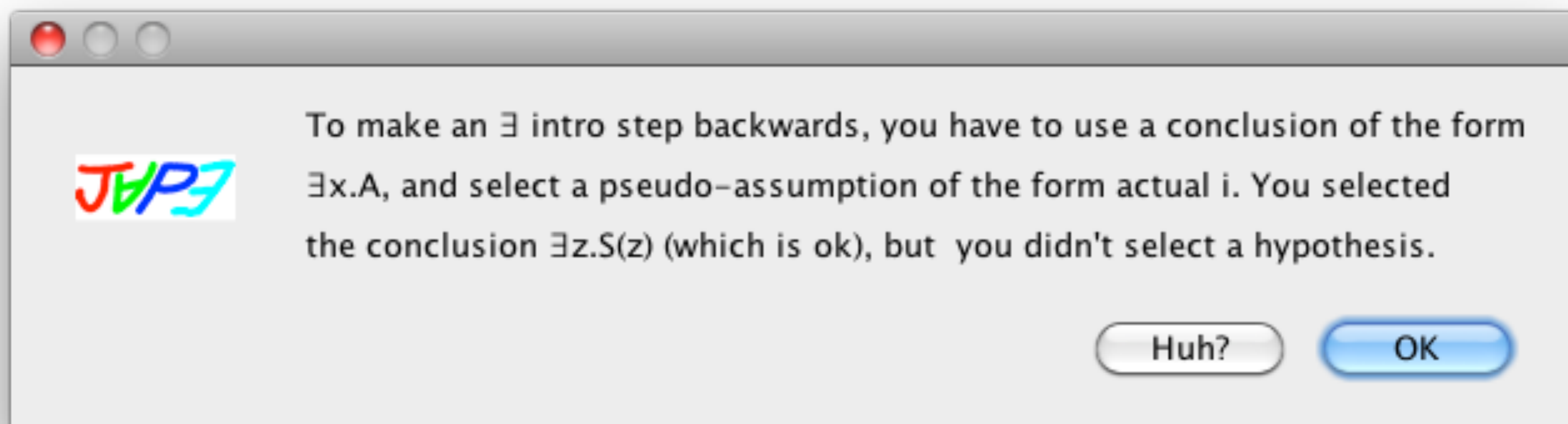
1: $\forall x.(R(x) \to S(x)),\ \exists y.R(y)$ premises

...

2: $\exists z.S(z)$

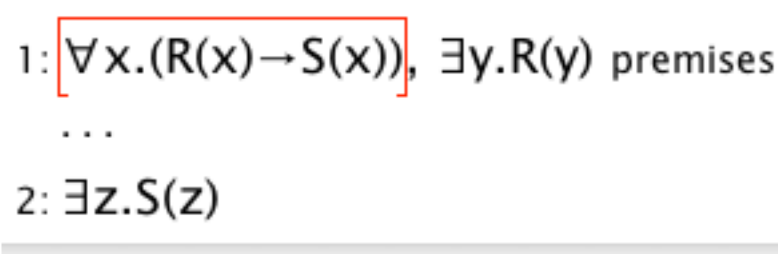


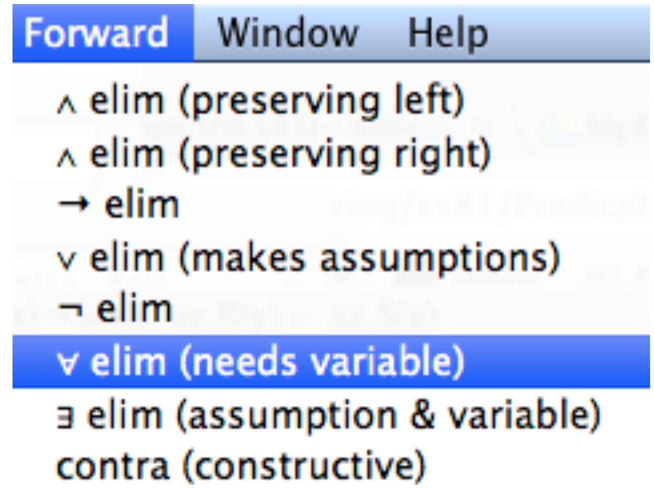
(“variable” means “term”)



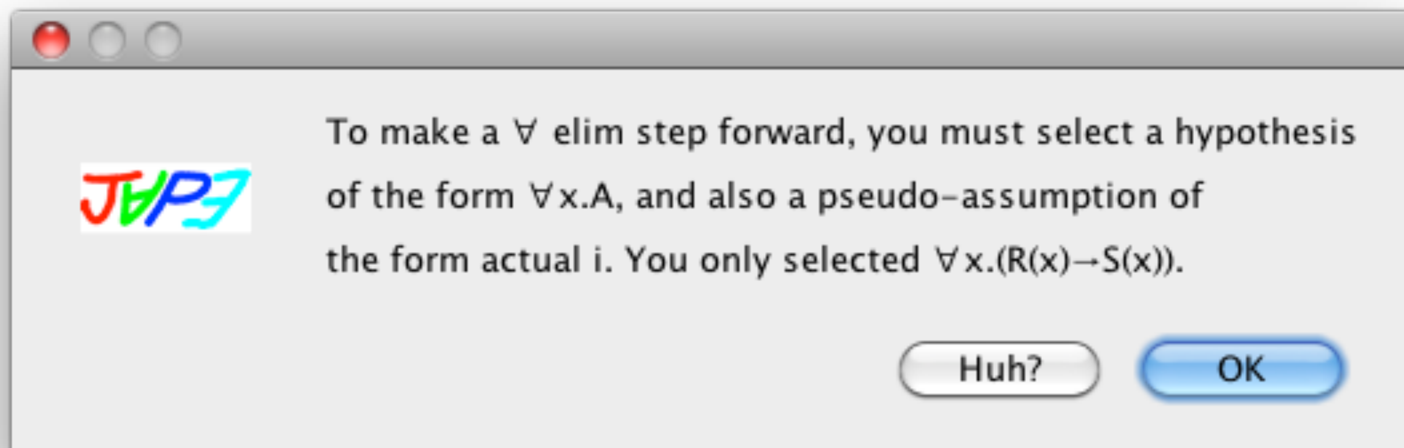
JAPE Example

- Cannot do this





("variable" means "term")



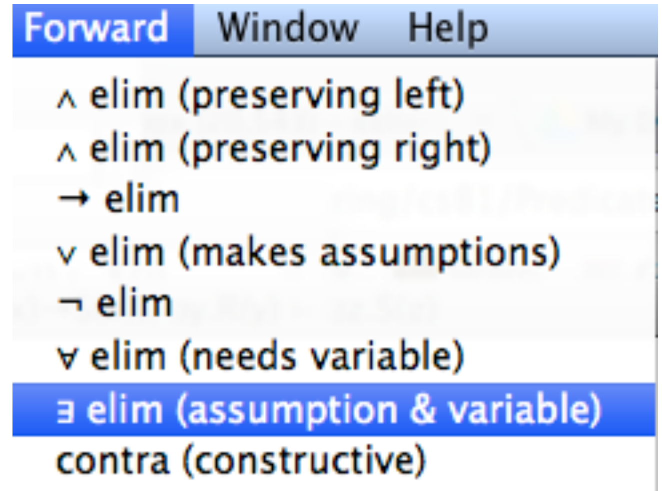
JAPE Example

- Start here

1: ∀x.(R(x)→S(x)), ∃y.R(y) premises

...

2: ∃z.S(z)



- Giving this:

1: ∀x.(R(x)→S(x)), ∃y.R(y) premises

2: actual i, R(i) assumptions

...

3: ∃z.S(z)

4: ∃z.S(z) ∃ elim 1.2,2-3

JAPE Example

- Now can use either ∃I or ∀E with variable i.

1: ∀x.(R(x)→S(x)), ∃y.R(y) premises

2: actual i, R(i) assumptions

...

3: ∃z.S(z)

4: ∃z.S(z) ∃ elim 1.2,2-3

1: ∀x.(R(x)→S(x)), ∃y.R(y) premises

2: actual i, R(i) assumptions

...

3: ∃z.S(z)

4: ∃z.S(z) ∃ elim 1.2,2-3

OR

1: ∀x.(R(x)→S(x)), ∃y.R(y) premises

2: actual i, R(i) assumptions

...

3: ∃z.S(z)

4: ∃z.S(z) ∃ elim 1.2,2-3

Next Steps

∃I **OR** ∀E

∃I

1:	∀x.(R(x)→S(x)), ∃y.R(y)	premises
2:	actual i, R(i)	assumptions
	. . .	
3:	S(i)	
4:	∃z.S(z)	∃ intro 3,2.1
5:	∃z.S(z)	∃ elim 1.2,2-4

∀E

1:	∀x.(R(x)→S(x)), ∃y.R(y)	premises
2:	actual i, R(i)	assumptions
3:	R(i)→S(i)	∀ elim 1.1,2.1
	. . .	
4:	∃z.S(z)	
5:	∃z.S(z)	∃ elim 1.2,2-4

Two steps to closure:

1:	∀x.(R(x)→S(x)), ∃y.R(y)	premises
2:	actual i, R(i)	assumptions
3:	R(i)→S(i)	∀ elim 1.1,2.1
4:	S(i)	→ elim 3,2.2
5:	∃z.S(z)	∃ intro 4,2.1
6:	∃z.S(z)	∃ elim 1.2,2-5

Other Examples

- I've added more examples to:

Proofs of Interest (Google Presentation)

Syllogisms

- A **syllogism** consists of three parts: the **major premise**, the **minor premise**, and the **conclusion**. In Aristotle, each of the premises is in the form "Some/all A belong to B," where "Some/All A" is one term and "belong to B" is another, but more modern logicians allow some variation. Each of the premises has one term in common with the conclusion: in a major premise, this is the major term (i.e., the predicate) of the conclusion; in a minor premise, it is the minor term (the subject) of the conclusion. For example:
  - **Major premise**: All humans are mortal.
  - **Minor premise**: Socrates is a human.
  - **Conclusion**: Socrates is mortal.
- Each of the three distinct terms represents a category, in this example, "human," "mortal," and "Socrates." "Mortal" is the major term; "Socrates," the minor term. The **premises also have one term in common** with each other, which is known as the **middle term** — in this example, "human."

Note: Stating a syllogism does not require validity.

Codifying Syllogisms using Predicate Logic

- Use unary predicates.
  - S(x): “x is an S”, “x has an S”, “x belongs to S”, etc.
- Use quantifiers for some, all
  - $\forall$ $\exists$
- Use connectives
  - $\neg$ $\rightarrow$
- Use constant symbols for individuals

Translating a Syllogism

Statement	Translation
All humans are mortal.	$\forall x\ (H(x) \rightarrow M(x))$
Socrates is a human.	$H(s)$
Therefore Socrates is mortal.	$M(s)$

This syllogism happens to be valid.

Syllogistic Forms

Statement Form	Translation
All S is/are/has... P.	$\forall x\ (S(x) \rightarrow P(x))$
Some S is P.	$\exists x\ (S(x) \wedge P(x))$
No S is P.	$\neg\exists x\ (S(x) \wedge P(x))$
Some S is not P.	$\exists x\ (S(x) \wedge \neg P(x))$
No S is not P.	$\neg\exists x\ (S(x) \wedge \neg P(x))$
All S is not P.	$\forall x\ (S(x) \rightarrow \neg P(x))$

Are any forms equivalent to one another?

Example: Translate this syllogism, then try to prove it.

- All fruit is nutritious.
- Some fruit is tasty.
- Therefore some tasty things are nutritious.

DeMorgan's Rules for Quantifiers

- Recall DeMorgan's rules for propositions
  - $(p \wedge q) \leftrightarrow \neg(\neg p \vee \neg q)$ $\quad$ $(\neg p \vee \neg q) \leftrightarrow \neg(p \wedge q)$
  - $(p \vee q) \leftrightarrow \neg(\neg p \wedge \neg q)$ $\quad$ $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$
- For quantifiers, we have analogous rules
  - $\forall x\, P(x) \leftrightarrow \neg(\exists x\, \neg P(x))$ $\quad$ $\exists x\, \neg P(x) \leftrightarrow \neg \forall x\, P(x)$
  - $\exists x\, P(x) \leftrightarrow \neg(\forall x\, \neg P(x))$ $\quad$ $\neg \exists x\, P(x) \leftrightarrow \forall x\, \neg P(x)$
- Note that in some cases, only one direction of implication is constructive.

Semantics of Predicate Logic

What is truth?

Interpretations of Formulas

- The **structure(s)** of interest in specific derivations are generally **not totally specified** in the system of derivation itself.

- Instead, we rely on certain formulas ("axioms") to **characterize** the properties of these structures.

- In natural deduction, these formulas will appear on the left-hand side of a sequent, or understood as lemmas.

- It can then be proved separately that the syntactic rules are in agreement with the semantics of the intended **interpretation**.

Interpretation I = (Δ, μ)

- An **interpretation** for a set of terms and formulas consists of:
  - A **domain** Δ (usually non-empty): contains all **individuals** of interest.
  - For each **constant symbol** c *in the language*, an element $\mu(c) \in \Delta$**.**
  - For each n-ary **function symbol** f, a function $\mu(f)\colon \Delta^n \rightarrow \Delta$.
  - For each n-ary **predicate symbol** p, a function $\mu(p)\colon \Delta^n \rightarrow \{T, F\}$.
- The values of μ are the values ***assigned*** by the interpretation.
- The domain, Δ, may also be called the "universe" or "domain of discourse".

Constant Symbols for a Domain

- In what follows, we will assume that there is a unique constant symbol for each domain element.
- The symbol will be **identified with** the element itself.
- Example: If $\Delta = \{1, 2, 3\}$, we will assume constant symbols 1, 2, 3.
- **This is only for sake of exposition. The symbols do not form a permanent part of the language.**

Truth of a Formula Relative to an Interpretation I = (Δ, μ)

- If a formula has **free variables**, add a ∀ quantifier for each such variable in front of the formula. (Free variables are understood to mean ∀-quantified by convention.)

- The result is called the "**closure**" of the formula.

- Now proceed **assuming the formula has no free variables**, i.e. the formula is **closed**.

- Our method will break down the formula recursively.

Truth of a Formula relative to an Interpretation I = (Δ, μ)

- For any closed formula φ, we will define the truth value $I[\varphi] \in \{T, F\}$.
- Brackets [ ] are used to emphasize that what is inside them is syntactic.

Truth of a Formula Relative to an Interpretation I = (Δ, μ)

- If the formula is of the **form ∀x** φ, then

  $I[\forall x\ \varphi] = T$ *iff* **for every** $d \in \Delta$: $I[\varphi[d/x]] = T$

- Recalling that we are using d as **constant** symbol in the case of substitution.

Truth of a Formula Relative to an Interpretation $I = (\Delta, \mu)$

- If the formula is of the **form ∃x** φ, then

  $I[\exists x\ \varphi] = T$ iff

  for *some* $d \in \Delta$: $I[\varphi[d/x]] = T$.

Truth of a Formula Relative to an Interpretation I = (Δ, μ)

- I[$\varphi \wedge \psi$] = T iff I[φ] = T **and** I[ψ] = T.

- (Recall there are no free variables.)

- I[$\varphi \vee \psi$] = T iff I[φ] = T **or** I[ψ] = T.

- Note that this is the same as the semantics for proposition logic.

Truth of a Formula Relative to an Interpretation $I = (\Delta, \mu)$

- $I[\varphi \rightarrow \psi] = T$ iff either $I[\varphi] = F$ or $I[\psi] = T$.

- $I[\varphi \leftrightarrow \psi] = T$ iff $I[\varphi] = I[\psi]$.

- $I[\neg\varphi] = T$ iff $I[\varphi] = F$.

- Finally, if the formula φ is **atomic**, then $I[\varphi]$ is determined according to the following slides.

The **value of terms** under an interpretation

- An interpretation $I = (\Delta, \mu)$ determines, for each **term** t, a value $I[t] \in \Delta$ of recursively:
  - If t is a **constant symbol** c, then $I[t] = \mu(c)$, the assigned value in Δ.
  - If t is a **function symbol** applied to terms, $f(t_1, t_2, \ldots, t_n)$ where the t_i are terms, then

    $$I[t] = \mu(f)(I[t_1], I[t_2], \ldots I[t_n])$$

    recalling that $\mu(f)$ is the **function** that interpretation I assigns the function symbol f.

The value of **atomic formulas** under an interpretation

- An interpretation $I = (\Delta, \mu)$ determines for each atomic formula E a value $I[E] \in \{T, F\}$ **recursively**:

  - If E is $p(t_1, t_2, \ldots, t_n)$, where p is an n-ary predicate symbol, and the t_i are its term arguments, then

    $$I[E] = \mu(p)(I[t_1], I[t_2], \ldots I[t_n]) \in \{T, F\}$$

    where $\mu(p)$ is the **predicate** I assigns to p.

    [using the **value** $I[t_i]$ **of terms** t_i presented previously]

Example

- Atomic Formula is: q(f(f(c)), c), where q is a predicate symbol, f is a function symbol, and c is a constant.

- *Suppose* interpretation I assigns
  - $\Delta = \{0, 1, 2\}$ domain
  - $\mu(c) = 0$ constant
  - $\mu(f) = \{0 \rightarrow 1, 1 \rightarrow 2, 2 \rightarrow 0\}$ function
  - $\mu(q) = \{(2, 0), (1, 2), (0, 1)\}$ predicate
    the set of pairs for which $\mu(q)$ is T
- Thus:
  - $I[f(f(c)] = \mu(f)(I[f(c)]) = \mu(f)(\mu(f)(I[c])) = \mu(f)(\mu(f)(\mu(c))) = 2$
  - $I(q(f(f(c)), c)) = \mu(q)(I[f(f(c)], \mu(c)) = \mu(q)(2, 0) = T$

Example

- Atomic Formula is: q(f(f(c)), f(c)), where q is a predicate symbol, f is a function symbol, and c is a constant.

- *Suppose* interpretation I assigns
  - $\Delta = \{0, 1, 2\}$ domain
  - $\mu(c) = 0$ constant
  - $\mu(f) = \{0 \rightarrow 1, 1 \rightarrow 2, 2 \rightarrow 0\}$ function
  - $\mu(q) = \{(2, 0), (1, 2), (0, 1)\}$ predicate
    the set of pairs for which $\mu(q)$ is T

- Thus:

  - $I[f(f(c)] = \mu(f)(I[f(c)]) = \mu(f)(\mu(f)(I[c])) = \mu(f)(\mu(f)(\mu(c))) = 2$

  - $I(q(f(f(c)), f(c))) = \mu(q)(I[f(f(c)], I[f(c)]) = \mu(q)(2, 1) = F$

Example

- Formula is: $\exists x\ q(f(f(c)), f(x))$, where q is a predicate symbol, f is a function symbol, and c is a constant.

- *Suppose* interpretation I assigns
  - $\Delta = \{0, 1, 2\}$ domain
  - $\mu(c) = 0$ constant
  - $\mu(f) = \{0 \rightarrow 1, 1 \rightarrow 2, 2 \rightarrow 0\}$ function
  - $\mu(q) = \{(2, 0), (1, 2), (0, 1))\}$ predicate
    the set of pairs for which $\mu(q)$ is T

- According to our rules $I[\exists x\ q(f(f(c)), f(x))] = T$ iff **at least one** of these is true:
  - $I[q(f(f(c)), f(0))]$ which is the same as $I[q(2, 1)]$
  - $I[q(f(f(c)), f(1))]$ which is the same as $I[q(2, 2)]$
  - $I[q(f(f(c)), f(2))]$ which is the same as $I[q(2, 0)]$

- As $q(2, 0) \in \mu(q)$, $I[\exists x\ q(f(f(c)), f(x))] = T$.

Example

- Formula is: ∀x q(f(f(c)), f(x)), where q is a predicate symbol, f is a function symbol, and c is a constant.

- *Suppose* interpretation I assigns
  - Δ = {0, 1, 2} domain
  - μ(c) = 0 constant
  - μ(f) = {0 → 1, 1 → 2, 2 → 0} function
  - μ(q) = {(2, 0), (1, 2), (0, 1))} predicate
    the set of pairs for which μ(q) is T

- According to our rules I[∃x q(f(f(c)), f(x))] = T iff **all** of these are true:
  - I[q(f(f(c)), f(0))] which is the same as I[q(2, 1)]
  - I[q(f(f(c)), f(1))] which is the same as I[q(2, 2)]
  - I[q(f(f(c)), f(2))] which is the same as I[q(2, 0)]

- As q(2, 1) is **not** in μ(q), I[∀x q(f(f(c)), f(x))] = F.

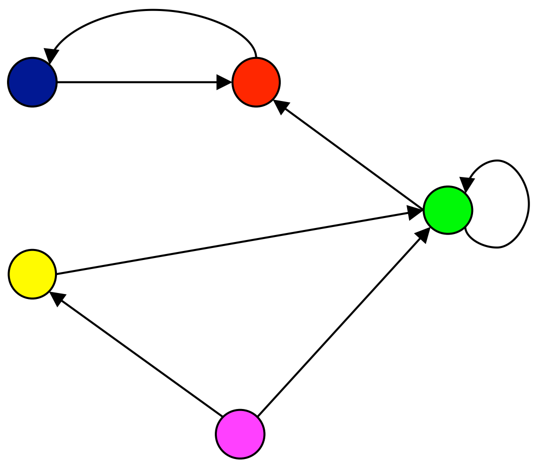
Fun with Interpretations

- A 2-ary predicate represents the binary relation in an interpretation, i.e. a set of pairs of domain elements.

- Various properties of relations can be expressed using predicate logic formulas.

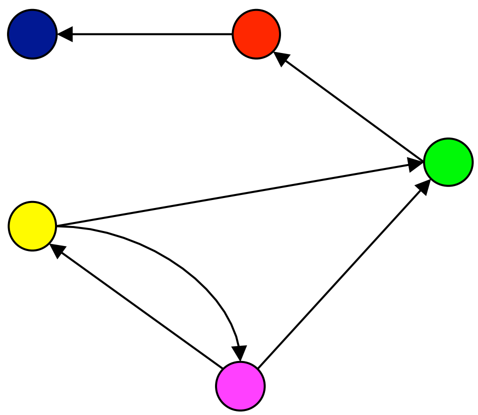
- In the following, what formula characterizes each relation represented by predicate L (sometimes using “loves” for analogy), and possibly the predicate =.

Everybody loves somebody.

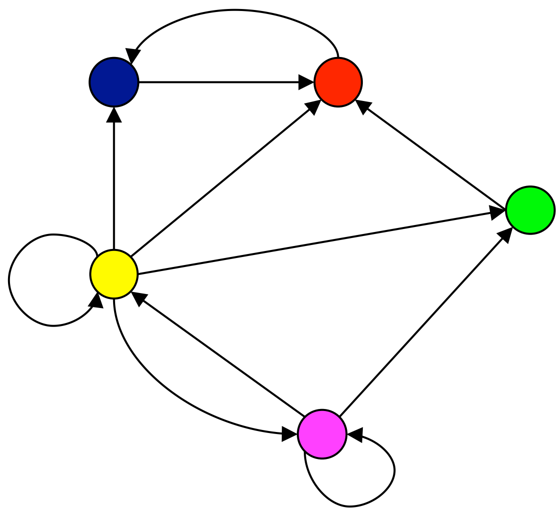


$\forall x \, \exists y \; L(x, y)$

Everybody is loved by somebody.

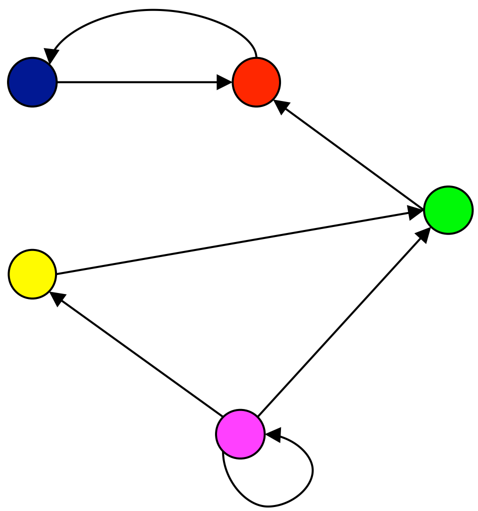


Somebody loves everybody.

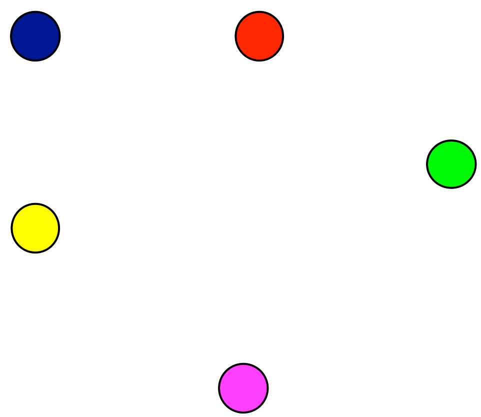


"Pollyanna"

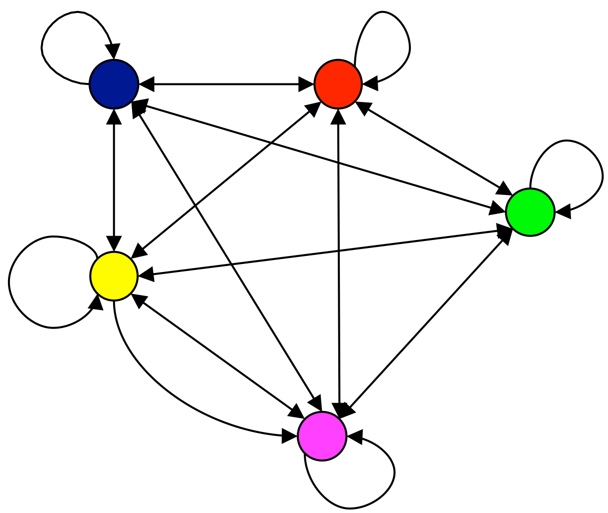
Nobody loves everybody.



Nobody loves somebody.

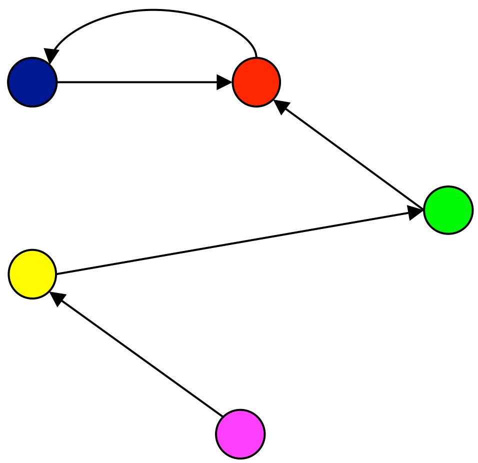


Everybody loves everybody.

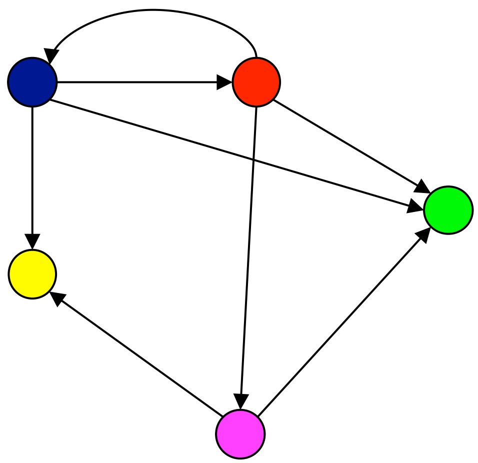


"Commune"

Everybody loves exactly one.

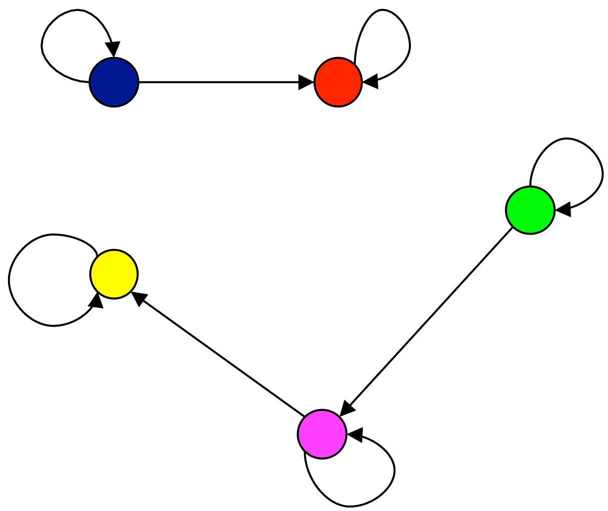


?



Everybody loves him/herself.

“Reflexive”



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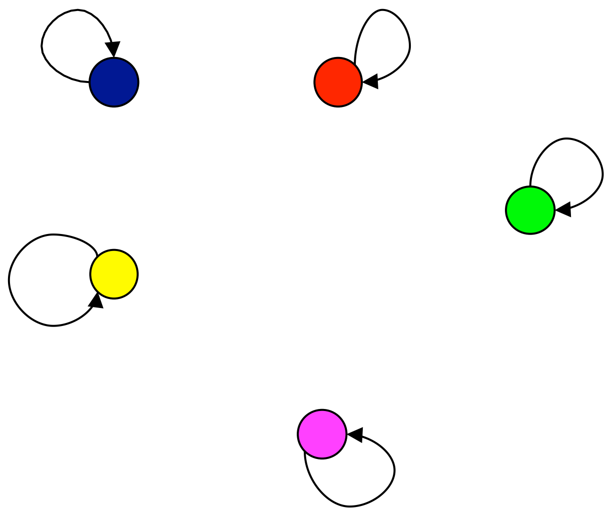
"Narcissists' Convention"



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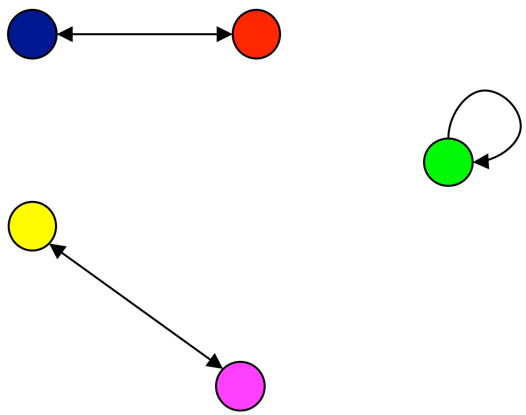
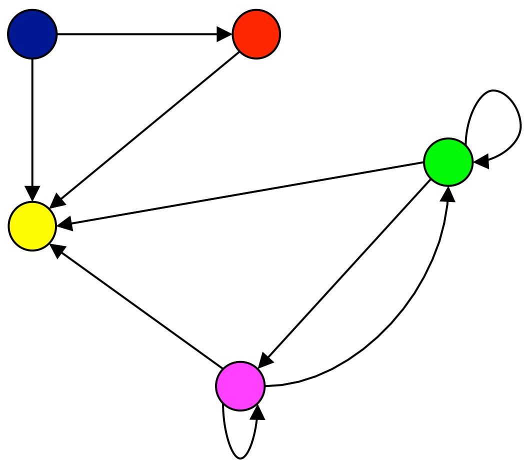
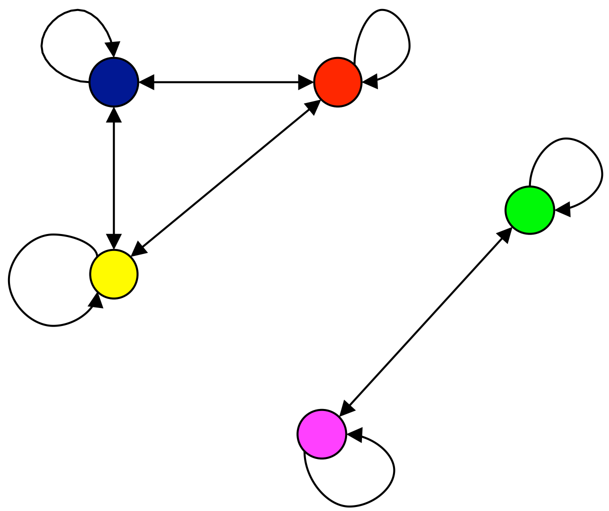


L is transitive.

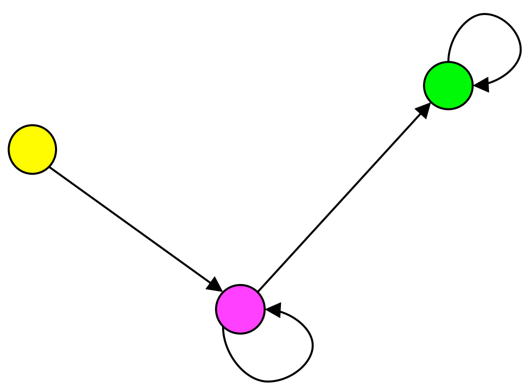


L is an equivalence relation

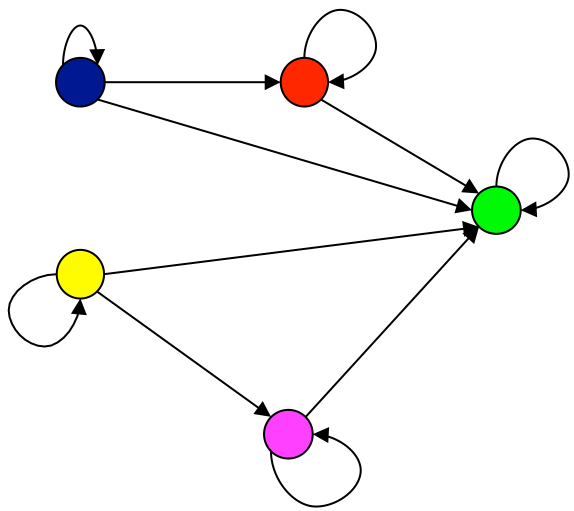


Reflexive, symmetric, transitive

L is antisymmetric.

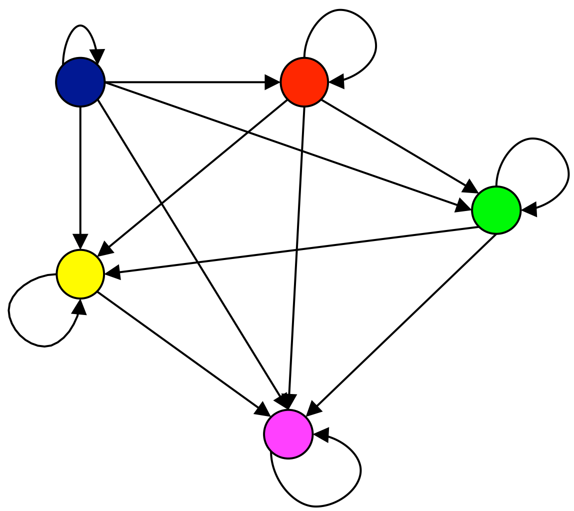


L is a partial order (“poset”).



Reflexive,
Antisymmetric,
Transitive

?



Empty Domains

- Customarily domains are required to be non-empty.
- Certain entailments that would be true under non-empty domains become false if the domain is empty.
- For example,

  $$\forall x\ P(x)\ \vert\!-\ \exists x\ P(x)$$

  The premise is *vacuously* true for an empty-domain, but the conclusion cannot be true.

We can construct a program for evaluting I [ ] **when the domain is finite**

- For infinite domains, this is not possible.

Satisfaction and Models

- An interpretation I ***satisfies*** a formula E iff I[E] = T.

- We also say that I ***is a model for*** E in this case.

- **Caution**: Some authors, such as Huth&Ryan, use “model” to mean “interpetation”.

- A formula is ***satisfiable*** iff there is an interpretation that satisfies it, otherwise it is ***unsatisfiable***.

Formalizing Semantic Entailment |=

- When $\varphi_1, \ldots, \varphi_n, \psi$ are predicate calculus formulas,

  $$\varphi_1, \ldots, \varphi_n \models \psi$$

  means:

  Every interpretation I that satisfies each of the formulas $\varphi_1, \ldots, \varphi_n$

  also satisfies ψ.

$\Gamma \models \psi$, where Γ is a ***set*** of formulas:

- Extend "**model for"** Γ to mean an interpretation satisfies the entire set Γ, as:

  Every model for Γ is also a model for ψ.

Validity

- When the left-hand side is empty:

  $$\models \psi$$

  we say that is **universally valid**,
  or just plain **valid**.

- In this case, every interpretation for ψ is a model.

- **Validity** in predicate calculus is analogous to **tautology** in propositional calculus.

|= in predicate calculus vs. propositional

- The predicate version of $|= \psi$ is a very broad statement:
  - The domain of an interpretation can be **infinite**.
  - The **set** of applicable interpretation is generally **infinite**.
- Intuitively there is much less likely to be an algorithm to check whether $|= \psi$ for predicate calculus in the way there is for the propositional calculus.

Predicate Calculus “with Equality”

- There is one exception to the “all interpretations” definitions of validity when the = predicate symbol is being used:

Equality is always interpreted as **identity** on the domain of the interpretation.

ND Equality Rules

- Natural Deduction typically introduces rules for equality (from which the axioms can be derived).
  - $$\frac{}{t = t}$$ =I where t is any term
  - $$\frac{s = t \qquad \varphi[s/x]}{\varphi[t/x]}$$ =E

    where s and t are any terms
    and x is any variable, provided
    s and t are free to replace x in φ.

Equality Formulas ("Axioms" in some systems) (*Derivable* from ND Rules)

- Four types of formulas characterize equality:
  - $\forall x\ (x = x)$ — reflexive
  - $\forall x\ \forall y\ (x = y \rightarrow y = x)$ — symmetric
  - $\forall x\ \forall y\ \forall z\ ((x = y \wedge y = z) \rightarrow (x = z))$ — transitive
  - $\forall x_1 \ldots \forall x_n\ \forall y_1 \ldots \forall y_n$
    $(x_1 = y_1 \wedge \ldots \wedge x_n = y_n) \rightarrow (f(x_1, \ldots x_n) = f(y_1, \ldots y_n))$ — substitution
    where f is any n-ary function symbol
  - $\forall x_1 \ldots \forall x_n\ \forall y_1 \ldots \forall y_n$
    $(x_1 = y_1 \wedge \ldots \wedge x_n = y_n) \rightarrow (p(x_1, \ldots x_n) \rightarrow p(y_1, \ldots y_n))$ — substitution
    where p is any n-ary predicate symbol

Example

$$\frac{}{t = t} \; =I$$

$$\frac{s = t \qquad \varphi[s/x]}{\varphi[t/x]} \; =E$$

Prove the symmetry rule:

$u = v \;\; |\!\!- \;\; v = u$

where u and v are any terms, from the two ND equality rules.

The "trick" here is finding the right φ.
Use x = u for φ, so that $\varphi[u/x]$ is u=u, an instance of =I.
This gives $\varphi[v/x]$ as v=u, the desired conclusion.

1. u = v	Premise	
2. u = u	=I	(identified as $\varphi[u/x]$)
3. v = u	1, 2, =E	(identified as $\varphi[v/x]$)

$$\frac{}{t = t}\ {=}\mathrm{I}$$

$$\frac{s = t \qquad \varphi[s/x]}{\varphi[t/x]}\ {=}\mathrm{E}$$

Example

Prove the transitivity rule:

$u = v, v = w \ \vdash\ u = w$

where u, v, and w are terms, from the two ND equality rules.

Here we **let** φ **be u = x**, to use =E rule.
Identify s = t in the =E rule with v = w.

1. $v = w$ — Premise
2. $u = v$ — Premise (identified as $\varphi[v/x]$)
3. $u = w$ — 1, 2, =E (identified as $\varphi[w/x]$)

Shortcutting Equality

- Having to spell out every instance of an equality rule can detract from the flow of a proof.

- So we will adopt the practice of combining uses of equality with other axioms, rules, and lemmas, knowing that we could spell them out if need be. (Be careful tho'!)

Theories

- In logic, a **theory** is a **set of formulas**.

- Usually a theory is determined by a set of **axioms**. The formulas in the theory are those derivable from the axioms using the rules of inference.

Examples of Theories

- **Group Theory** is the set of formulas derived from the group theory axioms:

$$\forall x\, \forall y\, \forall z\, [x + (y + z) = (x + y) + z],$$

$$\forall x\, [x + 0 = x],$$

$$\forall x\, \exists y\, [x + y = 0].$$

Here 0 is a constant symbol, **+** is a 2-ary function symbol,
= is a 2-ary predicate symbol.

An Alternate Version of Group Theory (adds inverse function – and symmetry)

- $\forall x \forall y \forall z\ (x+(y+z)) = ((x+y)+z)$ A1
- $\forall x\ (x+0 = x) \wedge (0+x = x)$ A2
- $\forall x\ (x+(-x) = 0) \wedge ((-x)+x = 0)$ A3

(-x is the y that exists, on the previous slide)

Note that symmetry does **not** say:

$$\forall x \forall y\ x + y = y + x$$

That does not follow from the group axioms.

Examples of Formulas Derivable in Group Theory

- $\forall x \; -(-x) = x$
- $\forall x \forall y \; -(x+y) = (-y)+(-x)$
- $\forall a \forall b \forall x \; (a+x = b) \rightarrow x = ((-a)+b)$

Sample Proof in Group Theory $\forall x\, -(-x) = x$ (Combining steps)

1.	$-x_0 + (-(-x_0)) = 0$	A3, ∀E, ∧E
2.	$x_0+((-x_0) + (-(-x_0))) = x_0 + 0$	1, =
3.	$(x_0+(-x_0)) + (-(-x_0)) = x_0 + 0$	2, A1
4.	$0 + (-(-x_0)) = x_0 + 0$	3, A2,=
5.	$0 + (-(-x_0)) = x_0$	4, A2, =
6.	$(-(-x_0)) = x_0$	5, A2, =
7.	$\forall x\, -(-x) = x$	1-6, ∀I

Examples of Theories

- **Abelian Groups** adds the commutative axiom to those of group theory:

$$\forall x \, \forall y \, [x + y = y + x]$$

- The set of provable formulas changes as a result.

Examples of Theories

- Theory of **Commutative Rings** adds a new function symbol, a new constant symbol, and more axioms to the theory of Abelian Groups:

$$\forall x \, \forall y \, [x \cdot y = y \cdot x],$$

$$\forall x \, \forall y \, \forall z \, [(x \cdot y) \cdot z = x \cdot (y \cdot z)],$$

$$\forall x \, \forall y \, \forall z \, [x \cdot (y + z) = (x \cdot y) + (x \cdot z)],$$

$$\forall x \, [x \cdot 1 = x],$$

$$0 \neq 1.$$

Limitations of Theories

- Some theories ***cannot*** be expressed with only a **finite** set of axioms in **first-order** logic (but can be in second-order).

- Example: Torsion-Free Abelian Groups adds an infinite number of axioms (one for each *n*, where *nx* means *x+x+…+x* *n* times): $\forall x\,[x \neq 0 \rightarrow nx \neq 0]$

Soundness and Completeness

- As with propositional logic, we define:
- **Soundness** of a set of derivation rules:

  For any set of formulas Γ and any formula ψ:

  $\Gamma \mid\!\!— \psi$ implies $\Gamma \mid\!= \psi$

- **Completeness** of a set of derivation rules:

  For any set of formulas Γ and any formula ψ:

  $\Gamma \mid\!= \psi$ implies $\Gamma \mid\!\!— \psi$

- It can be shown that our natural deduction framework has **both** of these properties [cf. van Dalen, *Logic and Structure*]

Examples of (Universally) Valid vs. Invalid Formulas

Invalid Formulas Valid Under Specific Interpretations

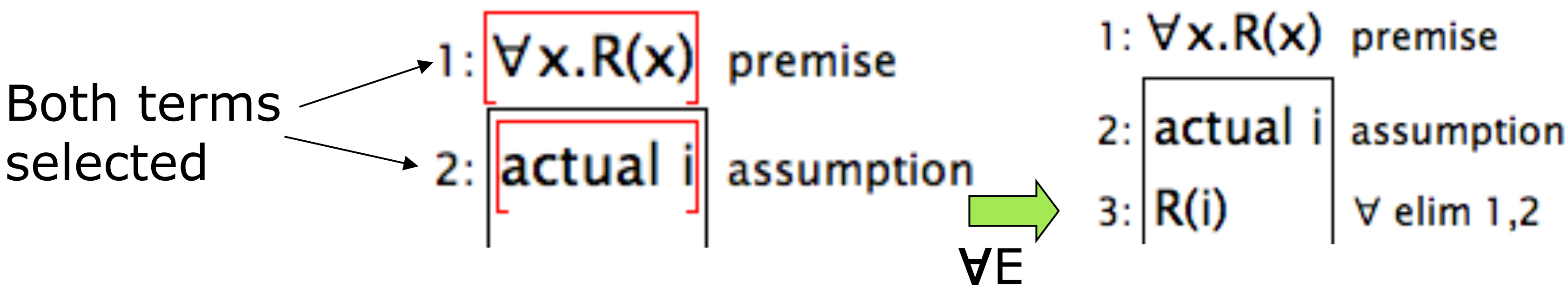
Showing a Formula Invalid

- Find a **counterexample**: an interpretation under which the formula is not valid.
- Example: $\forall x\, (A(x) \rightarrow B(x)) \rightarrow (\exists x\, A(x) \rightarrow \forall x\, B(x))$
- Interpretation:
  - $\Delta = \{1, 2\}$
  - $\mu(A) = \{2\}$
  - $\mu(B) = \{2\}$

More JAPE Examples

- **∀ Elimination** (working ***forward***) instantiates a ∀-quantified variable with **a term that already exists** (in this case, **i**).

- **Both** the term and the ∀ formula must be selected (using shift-click to add one or the other):



Note: If the red bracket opens **downward**, the item is usable as a hypothesis. If **upward**, a conclusion. In some cases both apply, and you need to click above or below to indicate which.

JAPE Examples

- ∀ Introduction (working backward), followed by ∀ Elimination (working forward)

1:	$\forall x.R(x)$	premise
2:	actual i	assumption
3:	$R(i)$	∀ elim 1,2
4:	$\forall y.R(y)$	∀ intro 2-3

- Note: JAPE will ***unify*** the above premise and conclusion, so a *shorter* proof, using the 'hyp' rule is, but this might be confusing because we end up with no y.

1: $\forall x.R(x)$ premise

...

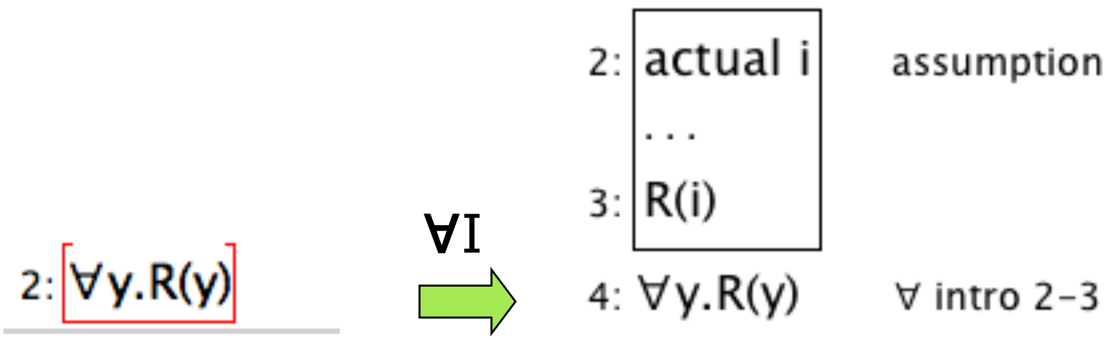
2: $\forall y.R(y)$

→ hyp

1: $\forall x.R(x)$ premise

JAPE Examples

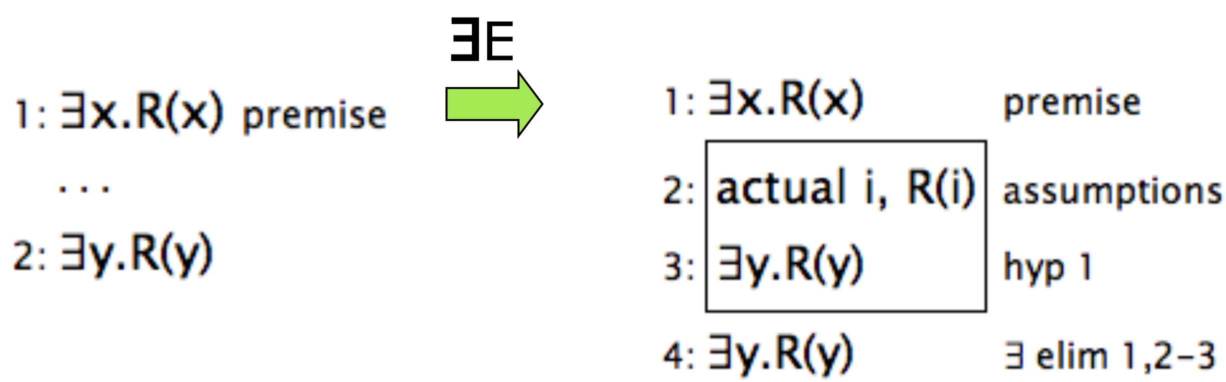
- **∀ Introduction** (working ***backward***) introduces a fresh variable. Variables are often helpful in completing a proof. Of course, the variable **can't be taken outside the box**.



i is meaningless out here

JAPE Examples

- **∃ Elimination** (working ***forward***) introduces a fresh variable for a sub-proof.
- It ***needs* a goal, in order to introduce the goal for the sub-proof** (inside the box). You may need to identify the goal if not obvious.
- In this example, the goal is implicit, and the proof is completed in one step.



Note: As before, JAPE will also *unify* the above premise and conclusion in a single step, making a proof unnecessary.

JAPE Examples

- **∃ Introduction** (working ***backward***) needs a term that it can use as an instantiation for the ∃ variable.
- The **JAPE ND theory doesn't have functions yet**, **so all such terms will be variables**.
- The variable must be selected by the user.
- We can't use ∃I here, because there is no variable available.

1: ∃x.R(x) premise
...
2: ∃y.R(y)

no variable

- Here is an example with a variable that ***can*** be used (but leads to a dead end):

2: actual i, ∃y.R(i,y) assumptions
...
3: ∃y.∃x.R(x,y)

∃I (with i for y)

2: actual i, ∃y.R(i,y) assumptions
...
3: ∃x.R(x,i)
4: ∃y.∃x.R(x,y) ∃ intro 3,2.1

Proof of a sequent using ∃E and ∃I

Work forward to introduce variables, by **eliminating** ∃'s (opening boxes):

1: ∃x.∃y.R(x,y) premise
. . .
2: ∃y.∃x.R(x,y)

∃E →

1: ∃x.∃y.R(x,y) premise
2: actual i, ∃y.R(i,y) assumptions
. . .
3: ∃y.∃x.R(x,y)
4: ∃y.∃x.R(x,y) ∃ elim 1,2–3

∃E →

1: ∃x.∃y.R(x,y) premise
2: actual i, ∃y.R(i,y) assumptions
3: actual i1, R(i,i1) assumptions
. . .
4: ∃y.∃x.R(x,y)
5: ∃y.∃x.R(x,y) ∃ elim 2.2,3–4
6: ∃y.∃x.R(x,y) ∃ elim 1,2–5

(Note the conclusions inside = the conclusion outside.)

then introduce ∃'s working backward **in the right order**:

1: ∃x.∃y.R(x,y) premise
2: actual i, ∃y.R(i,y) assumptions
3: actual i1, R(i,i1) assumptions
. . .
4: ∃y.∃x.R(x,y)
5: ∃y.∃x.R(x,y) ∃ elim 2.2,3–4
6: ∃y.∃x.R(x,y) ∃ elim 1,2–5

identify y & i1

∃I →

1: ∃x.∃y.R(x,y) premise
2: actual i, ∃y.R(i,y) assumptions
3: actual i1, R(i,i1) assumptions
. . .
4: ∃x.R(x,i1)
5: ∃y.∃x.R(x,y) ∃ intro 4,3.1
6: ∃y.∃x.R(x,y) ∃ elim 2.2,3–5
7: ∃y.∃x.R(x,y) ∃ elim 1,2–6

identify x & i

∃I →

1: ∃x.∃y.R(x,y) premise
2: actual i, ∃y.R(i,y) assumptions
3: actual i1, R(i,i1) assumptions
4: ∃x.R(x,i1) ∃ intro 3.2,2.1
5: ∃y.∃x.R(x,y) ∃ intro 4,3.1
6: ∃y.∃x.R(x,y) ∃ elim 2.2,3–5
7: ∃y.∃x.R(x,y) ∃ elim 1,2–6

Example using all four quantifier rules

1:	$\exists x.\forall y.R(x,y)$	premise
	...	
2:	$\forall y.\exists x.R(x,y)$	

$\forall I$ →

1:	$\exists x.\forall y.R(x,y)$	premise
2:	actual i	assumption
	...	
3:	$\exists x.R(x,i)$	
4:	$\forall y.\exists x.R(x,y)$	$\forall$ intro 2–3

$\exists E$ →

1:	$\exists x.\forall y.R(x,y)$	premise
2:	actual i	assumption
3:	actual i1, $\forall y.R(i1,y)$	assumptions
	...	
4:	$\exists x.R(x,i)$	
5:	$\exists x.R(x,i)$	$\exists$ elim 1,3–4
6:	$\forall y.\exists x.R(x,y)$	$\forall$ intro 2–5

$\exists I$ →

1:	$\exists x.\forall y.R(x,y)$	premise
2:	actual i	assumption
3:	actual i1, $\forall y.R(i1,y)$	assumptions
	...	
4:	$R(i1,i)$	
5:	$\exists x.R(x,i)$	$\exists$ intro 4,3.1
6:	$\exists x.R(x,i)$	$\exists$ elim 1,3–5
7:	$\forall y.\exists x.R(x,y)$	$\forall$ intro 2–6

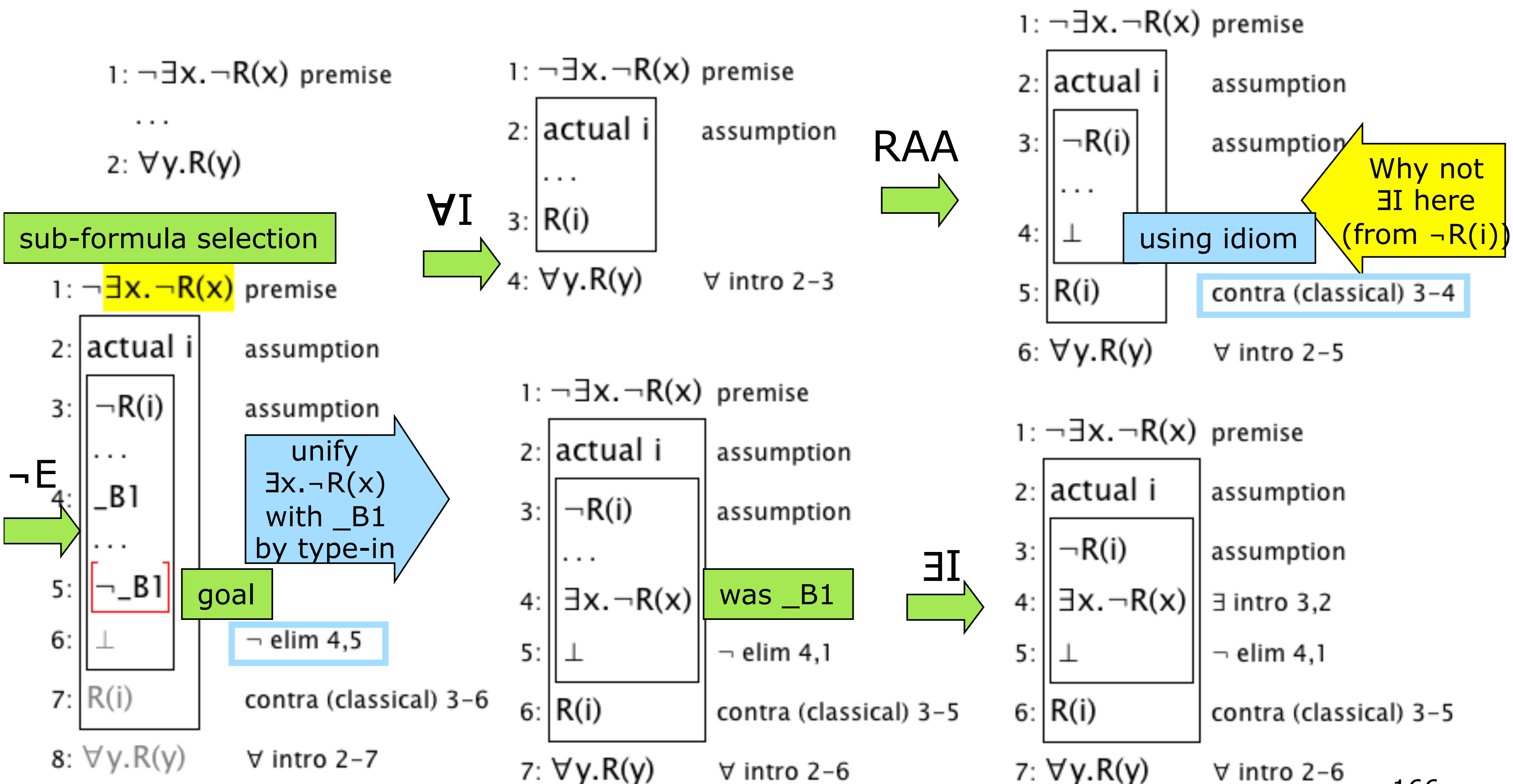
$\forall E$ →

1:	$\exists x.\forall y.R(x,y)$	premise
2:	actual i	assumption
3:	actual i1, $\forall y.R(i1,y)$	assumptions
4:	$R(i1,i)$	$\forall$ elim 3.2,2
5:	$\exists x.R(x,i)$	$\exists$ intro 4,3.1
6:	$\exists x.R(x,i)$	$\exists$ elim 1,3–5
7:	$\forall y.\exists x.R(x,y)$	$\forall$ intro 2–6

A Common Idiom in JAPE

- These steps are often done in sequence, but it is not always obvious when to use them:
  - contra (classical) = RAA
  - ¬ elimination (introduces skeletal formulas)
  - unify one of the skeletal formulas with an existing sub-formula

Sometimes the steps have to be taken in a round-about order, e.g. ∃I won't work forward (needs variable and body).
This example uses the previous idiom.



JAPE

- **The non-empty universe assumption is not assumed in JAPE!!**
- If you need this, you must introduce a premise that there is at least one element. How to do this is shown on the next slide.
- Proved in textbooks, but not provable in JAPE:

1: ∀x.R(x) premise

...

2: ∃x.R(x)

- Can't go backward, because ∃I needs a term.
- Can't go forward, because ∀E needs a variable.

∀ intro (introduces variable)
∃ intro (needs variable)
∀ elim (needs variable)
∃ elim (assumption & variable)

JAPE

- If you **need** the non-empty universe assumption, you must introduce a premise that **there is at least one element**, by including 'actual i', or '∃x.T' as a premise. (one place where T is useful, but others could be used).

1:	actual i, $\forall x.R(x)$	premises
2:	$R(i)$	$\forall$ elim 1.2,1.1
3:	$\exists x.R(x)$	$\exists$ intro 2,1.1

1:	$\exists x.\top$, $\forall x.R(x)$	premises
2:	actual i, $\top$	assumptions
3:	$R(i)$	$\forall$ elim 1.2,2.1
4:	$\exists y.R(y)$	$\exists$ intro 3,2.1
5:	$\exists y.R(y)$	$\exists$ elim 1.1,2-4

- See Bornat's book "Proof and Disproof … " for discussion on why this philosophy is better.

A Tricky One

1: actual j, actual k   premises

...

2: $\exists x.(R(x) \rightarrow R(j) \wedge R(k))$

A Tricky One

1: actual j, actual k — premises

...

2: $\exists x.(R(x) \rightarrow R(j) \wedge R(k))$

Note: This does ***not*** say that j and k are ***distinct***. They could be two names for the same individual.

1: actual j, actual k — premises

2: $\_E \vee \neg \_E$ — Theorem $E \vee \neg E$

...

3: $\exists x.(R(x) \rightarrow R(j) \wedge R(k))$

LEM to the rescue (used as a lemma)

unify R(j) with _E

1: actual j, actual k — premises

2: $R(j) \vee \neg R(j)$ — Theorem $E \vee \neg E$

...

3: $\exists x.(R(x) \rightarrow R(j) \wedge R(k))$

$\vee$E

1: actual j, actual k — premises

2: $R(j) \vee \neg R(j)$ — Theorem $E \vee \neg E$

3: $R(j)$ — assumption

...

4: $\exists x.(R(x) \rightarrow R(j) \wedge R(k))$

What x would make this work?

5: $\neg R(j)$ — assumption

...

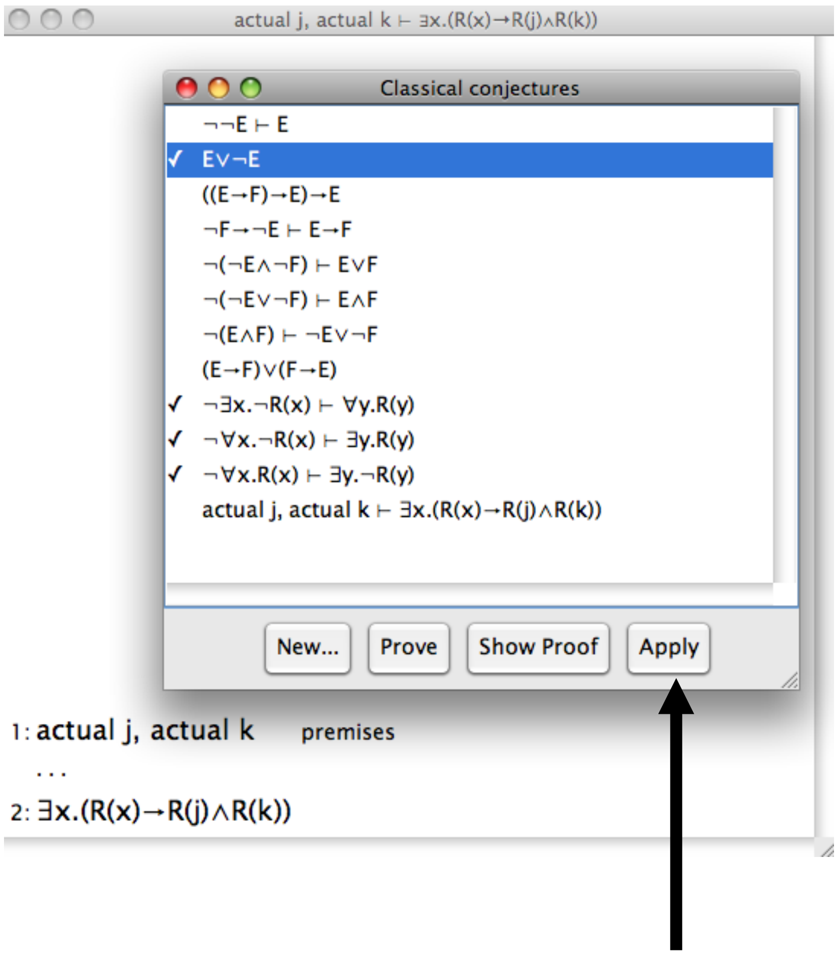
6: $\exists x.(R(x) \rightarrow R(j) \wedge R(k))$

What x would make this work?

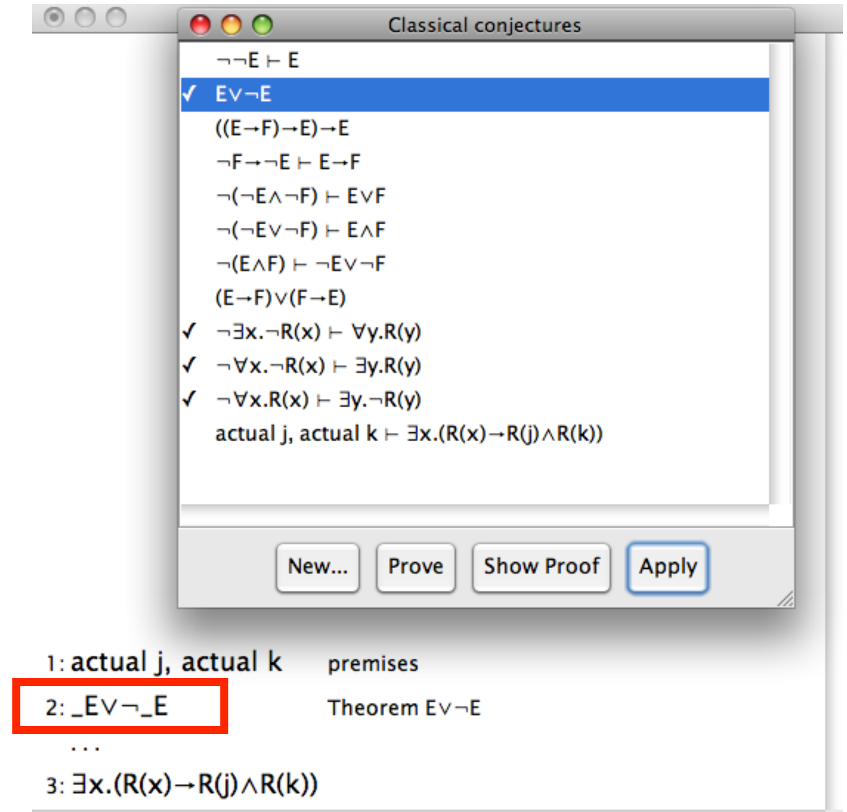
7: $\exists x.(R(x) \rightarrow R(j) \wedge R(k))$ — $\vee$ elim 2,3–4,5–6

How to introduce LEM (it must be proved first)

1:	¬(E∨¬E)	assumption
2:	E	assumption
3:	E∨¬E	∨ intro 2
4:	⊥	¬ elim 3,1
5:	¬E	¬ intro 2–4
6:	E∨¬E	∨ intro 5
7:	⊥	¬ elim 6,1
8:	E∨¬E	contra (classical) 1–7



Click to apply as lemma



Voila!

Continuing the tricky proof …

For the Top Box

x = k (an actual) will enable ∃I

3:	$R(j)$	assumption
	. . .	
4:	$\exists x.(R(x) \rightarrow R(j) \land R(k))$	

3:	$R(j)$	assumption
4:	$R(k)$	assumption
5:	$R(j) \land R(k)$	∧ intro 3,4
6:	$R(k) \rightarrow R(j) \land R(k)$	→ intro 4–5
7:	$\exists x.(R(x) \rightarrow R(j) \land R(k))$	∃ intro 6,1.2

Continuing the tricky proof …

For the Bottom Box

x = j (an actual) will enable ∃I (using contra)

5:	$\neg R(j)$	assumption
	. . .	
6:	$\exists x.(R(x) \rightarrow R(j) \wedge R(k))$	

8:	$\neg R(j)$	assumption
9:	$R(j)$	assumption
10:	$\perp$	¬ elim 9,8
11:	$R(j) \wedge R(k)$	contra (constructive) 10
12:	$R(j) \rightarrow R(j) \wedge R(k)$	→ intro 9–11
13:	$\exists x.(R(x) \rightarrow R(j) \wedge R(k))$	∃ intro 12,1.1

Completed Proof

1:	actual j, actual k	premises
2:	$R(j) \vee \neg R(j)$	Theorem $E \vee \neg E$
3:	$R(j)$	assumption
4:	$R(k)$	assumption
5:	$R(j) \wedge R(k)$	$\wedge$ intro 3,4
6:	$R(k) \rightarrow R(j) \wedge R(k)$	$\rightarrow$ intro 4-5
7:	$\exists x.(R(x) \rightarrow R(j) \wedge R(k))$	$\exists$ intro 6,1.2
8:	$\neg R(j)$	assumption
9:	$R(j)$	assumption
10:	$\bot$	$\neg$ elim 9,8
11:	$R(j) \wedge R(k)$	contra (constructive) 10
12:	$R(j) \rightarrow R(j) \wedge R(k)$	$\rightarrow$ intro 9-11
13:	$\exists x.(R(x) \rightarrow R(j) \wedge R(k))$	$\exists$ intro 12,1.1
14:	$\exists x.(R(x) \rightarrow R(j) \wedge R(k))$	$\vee$ elim 2,3-7,8-13

An Analogous Sequent

1: actual i premise

...

2: $\exists x.(R(x) \rightarrow \forall y.R(y))$

"If there is at least one person, then there is someone (x) such that if x is happy then everyone is happy."

Key

- How to use the LEM to create a dichotomy?
- $E \vee \neg E$
- But what is E?

1: actual i   premise

...

2: $\exists x.(R(x) \rightarrow \forall y.R(y))$

- Possibilities for E:
  - $\exists x.R(x)$
  - $\forall y.R(y)$

- Use unification to assign formula to E

Constructive → vs. Classical ←

1:	$E \wedge F$	premise
2:	$\neg E \vee \neg F$	assumption
3:	$\neg E$	assumption
4:	E	∧ elim 1
5:	$\bot$	¬ elim 4,3
6:	$\neg F$	assumption
7:	F	∧ elim 1
8:	$\bot$	¬ elim 7,6
9:	$\bot$	∨ elim 2,3–5,6–8
10:	$\neg(\neg E \vee \neg F)$	¬ intro 2–9

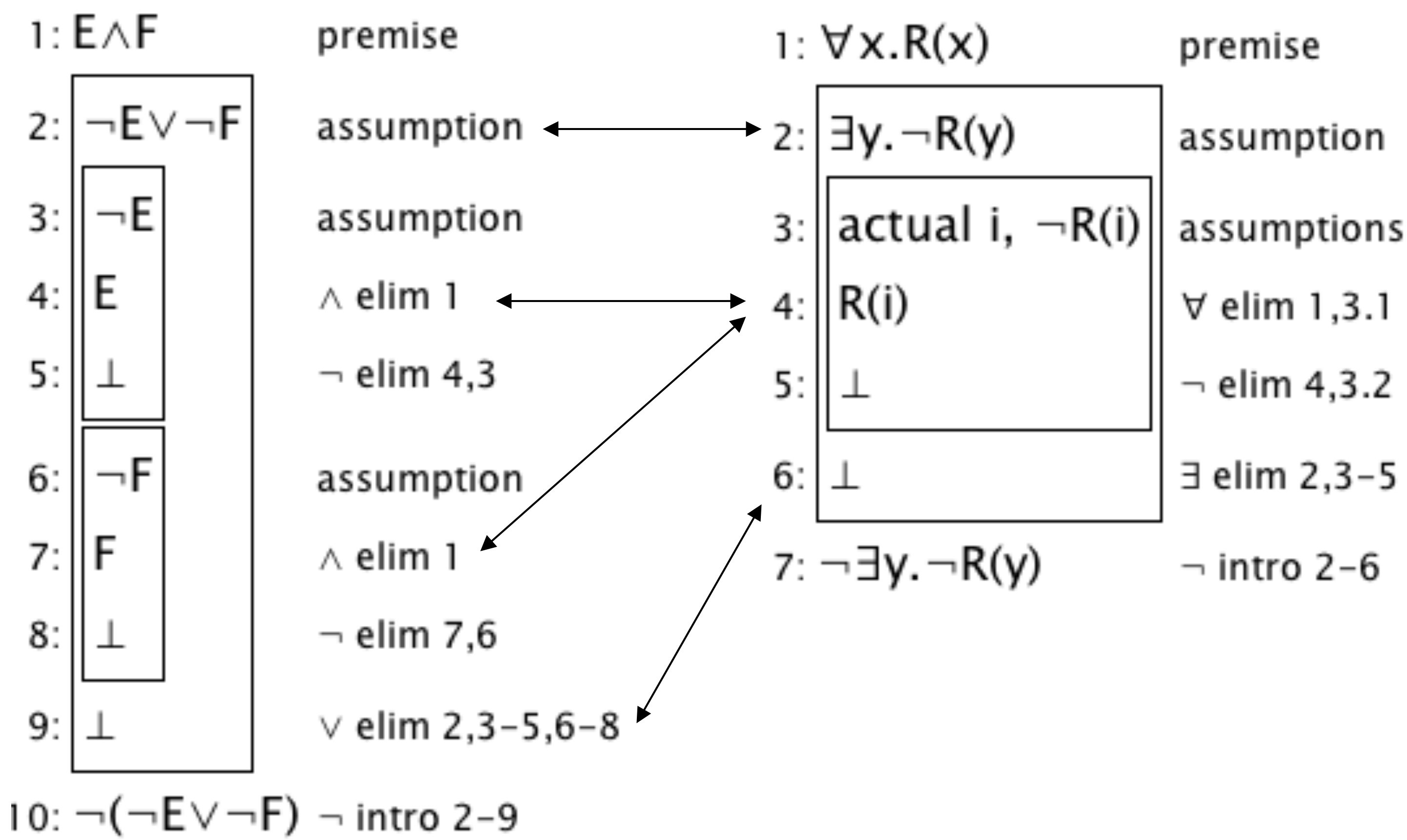
1:	$\neg(\neg E \vee \neg F)$	premise
2:	$\neg E$	assumption
3:	$\neg E \vee \neg F$	∨ intro 2
4:	$\bot$	¬ elim 3,1
5:	E	contra (classical) 2–4
6:	$\neg F$	assumption
7:	$\neg E \vee \neg F$	∨ intro 6
8:	$\bot$	¬ elim 7,1
9:	F	contra (classical) 6–8
10:	$E \wedge F$	∧ intro 5,9

Constructive → vs. Classical ←

1:	$\forall x.R(x)$	premise
2:	$\exists y.\neg R(y)$	assumption
3:	actual i, $\neg R(i)$	assumptions
4:	$R(i)$	$\forall$ elim 1,3.1
5:	$\perp$	$\neg$ elim 4,3.2
6:	$\perp$	$\exists$ elim 2,3–5
7:	$\neg\exists y.\neg R(y)$	$\neg$ intro 2–6

1:	$\neg\exists x.\neg R(x)$	premise
2:	actual i	assumption
3:	$\neg R(i)$	assumption
4:	$\exists x.\neg R(x)$	$\exists$ intro 3,2
5:	$\perp$	$\neg$ elim 4,1
6:	$R(i)$	contra (classical) 3–5
7:	$\forall y.R(y)$	$\forall$ intro 2–6

Note Rule Parallels



Note Rule Parallels

1:	$\neg(\neg E \vee \neg F)$	premise
2:	$\neg E$	assumption
3:	$\neg E \vee \neg F$	$\vee$ intro 2
4:	$\bot$	$\neg$ elim 3,1
5:	E	contra (classical) 2–4
6:	$\neg F$	assumption
7:	$\neg E \vee \neg F$	$\vee$ intro 6
8:	$\bot$	$\neg$ elim 7,1
9:	F	contra (classical) 6–8
10:	$E \wedge F$	$\wedge$ intro 5,9

1:	$\neg\exists x.\neg R(x)$	premise
2:	actual i	assumption
3:	$\neg R(i)$	assumption
4:	$\exists x.\neg R(x)$	$\exists$ intro 3,2
5:	$\bot$	$\neg$ elim 4,1
6:	$R(i)$	contra (classical) 3–5
7:	$\forall y.R(y)$	$\forall$ intro 2–6

A proof

- Suppose everyone loves somebody, and loves is symmetric and transitive.
- Then loves is reflexive.

1:	∀x.∃y.R(x,y), ∀x.∀y.(R(x,y)→R(y,x))	premises
2:	∀x.∀y.∀z.((R(x,y)∧R(y,z))→R(x,z))	premise
3:	actual i	assumption
4:	∀y.∀z.((R(i,y)∧R(y,z))→R(i,z))	∀ elim 2,3
5:	∀y.(R(i,y)→R(y,i))	∀ elim 1.2,3
6:	∃y.R(i,y)	∀ elim 1.1,3
7:	actual i1	assumption
8:	R(i,i1)	assumption
9:	∀z.((R(i,i1)∧R(i1,z))→R(i,z))	∀ elim 4,7
10:	(R(i,i1)∧R(i1,i))→R(i,i)	∀ elim 9,3
11:	R(i,i1)→R(i1,i)	∀ elim 5,7
12:	R(i1,i)	→ elim 11,8
13:	R(i,i1)∧R(i1,i)	∧ intro 8,12
14:	R(i,i)	→ elim 10,13
15:	R(i,i)	∃ elim 6,7-14
16:	∀x.R(x,x)	∀ intro 3-15

Informal Proof

- Assertion: If everyone loves somebody, and loves is symmetric and transitive, then loves is reflexive.

- Let x_0 be an arbitrary element, to show x_0 loves x_0.

- Since everyone loves someone, let y_0 be someone x_0 loves.

- By symmetry, y_0 loves x_0 too.

- By transitivity, since x_0 loves and y_0 loves x_0 , x_0 loves x_0.

Caution

- Without the assumption:
  Everyone loves somebody
- the assertion that loves is reflexive does not hold.

- Show this by giving a counterexample.

How to do without function symbols

- Every n-ary function is an (n+1)-ary relation.
- For example, a binary function f can be represented by a 3-ary relation F.
- F(x, y, z) means f(x, y) = z.
- Functionality induces some additional axioms for F:
  - $\forall x\ \forall y\ \exists z\ F(x, y, z)$
  - $\forall x\ \forall y\ \forall z\ \forall z'\ (F(x, y, z) \wedge F(x, y, z') \rightarrow z = z')$
- We'd still need axioms for equality.

Example: Group theory without function symbols (e is identity)

- $\forall x \, \forall y \, \exists z \, F(x, y, z)$
- $\forall x \, \forall y \, \forall z \, \forall z' \, (F(x, y, z) \wedge F(x, y, z') \rightarrow z = z')$
- $\forall x \, \forall y \, \forall z \, \exists v(F(x, y, v) \wedge F(v, z, w)) \rightarrow$
  $\exists u(F(y, z, u) \wedge F(x, u, w))$
- $\forall x \, F(x, e, x)$
- $\forall x \, F(e, x, x)$
- $\forall x \, \exists y \, F(x, y, e)$
- + Equality axioms