Tableau and Sequent Calculus for Predicate Logic

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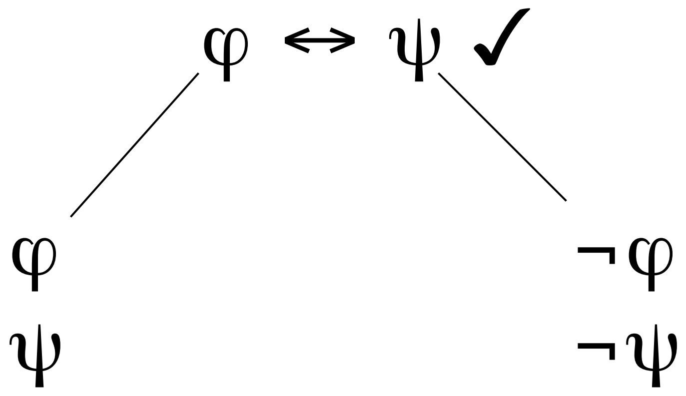
Use of Tableaux

Although tableaux might not see uses in published math proofs, it is a useful way to check the validity of a sequent.

Aside: Additional Tableau Rules: $\varphi \leftrightarrow \psi$

- This formula is retired and the tree **splits** with the formulas and their negations **stacked**:



$\varphi \leftrightarrow \psi$ ✓

φ

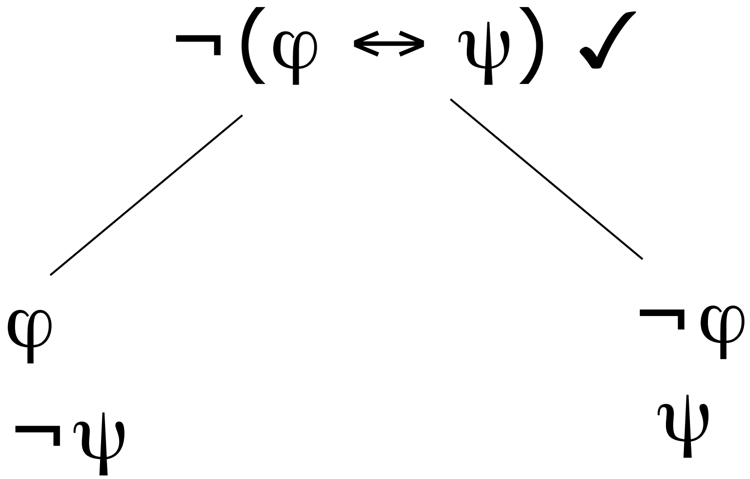
ψ

$\neg\varphi$

$\neg\psi$

Additional Tableau Rules: $\neg(\varphi \leftrightarrow \psi)$

- This formula is retired and the tree **splits**, with complementary formulas **stacked**:



$\neg(\varphi \leftrightarrow \psi)$ ✓

φ

$\neg\psi$

$\neg\varphi$

ψ

Propositional Tableau Rule Summary

stack	$\varphi \wedge \psi$ φ ψ	$\neg(\varphi \vee \psi)$ $\neg\varphi$ $\neg\psi$	$\neg(\varphi \rightarrow \psi)$ φ $\neg\psi$		$\neg\neg\varphi$ φ
split	$\varphi \vee \psi$ $\varphi \quad \psi$	$\neg(\varphi \wedge \psi)$ $\neg\varphi \quad \neg\psi$	$\varphi \rightarrow \psi$ $\neg\varphi \quad \psi$	$(\varphi \leftrightarrow \psi)$ $\varphi \quad \neg\varphi$ $\psi \quad \neg\psi$	
				$\neg(\varphi \leftrightarrow \psi)$ $\varphi \quad \neg\varphi$ $\neg\psi \quad \psi$	

Quantifier Rules for Tableaux

¬∃ rule stacks

$\neg\exists\nu\ \varphi$ ✓

$\forall\nu\ \neg\varphi$

¬∀ rule stacks

$\neg \forall \nu\ \varphi$ ✓

$\exists \nu\ \neg \varphi$

∃ rule stacks

$$\begin{array}{c} \exists \nu \ \varphi \ \checkmark \\ \varphi[c/\nu] \end{array}$$

where c is a **new** constant not appearing in the tree.

This rule can be used only **once per path** per ∃ formula.

It can be used multiply on separate paths however.

∀ rule

$\forall \nu\ \varphi$ **Does not get a check!!**

$\varphi[\tau/\nu]$

where τ is any term free to replace ν in φ.

This rule can used arbitrarily-many times for a ∀ formula.

It is the “gift that keeps on giving”.

Example: Prove ∃x ∀y p(x, y) |– ∀y∃x p(x, y)

1.	∃x ∀y p(x, y)	LHS
2.	¬∀y∃x p(x, y)	RHS negated

Example: Prove ∃x ∀y p(x, y) |– ∀y∃x p(x, y)

1.	∃x ∀y p(x, y)		LHS
2.	¬∀y∃x p(x, y)	✓	RHS negated
3.	∃y¬∃x p(x, y)		2, ¬∀ rule

Example: Prove $\exists x\ \forall y\ p(x, y)$ |– $\forall y \exists x\ p(x, y)$

1.	$\exists x\ \forall y\ p(x, y)$	✓	LHS
2.	$\neg \forall y \exists x\ p(x, y)$	✓	RHS negated
3.	$\exists y \neg \exists x\ p(x, y)$		2, $\neg\forall$ rule
4.	$\forall y\ p(a, y)$		1, $\exists$ rule

Example: Prove $\exists x\, \forall y\, p(x, y) \mid\!-\ \forall y \exists x\, p(x, y)$

1.	$\exists x\, \forall y\, p(x, y)$	✓	LHS
2.	$\neg\forall y \exists x\, p(x, y)$	✓	RHS negated
3.	$\exists y \neg\exists x\, p(x, y)$	✓	2, $\neg\forall$ rule
4.	$\forall y\, p(a, y)$		1, $\exists$ rule
5.	$\neg\exists x\, p(x, b)$		3, $\exists$ rule
6.	$p(a, b)$		4, $\forall$ rule

Example: Prove $\exists x\ \forall y\ p(x, y) \mid\!- \forall y \exists x\ p(x, y)$

1.	$\exists x\ \forall y\ p(x, y)$	✓	LHS
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3.	$\exists y \neg \exists x\ p(x, y)$	✓	2, $\neg\forall$ rule
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5.	$\neg \exists x\ p(x, b)$	✓	3, $\exists$ rule
6.	$p(a, b)$		4, $\forall$ rule
7.	$\forall x\ \neg p(x, b)$		5, $\neg\exists$ rule

Example: Prove $\exists x\, \forall y\, p(x, y) \mid\!- \forall y \exists x\, p(x, y)$

1.	$\exists x\, \forall y\, p(x, y)$	✓	LHS
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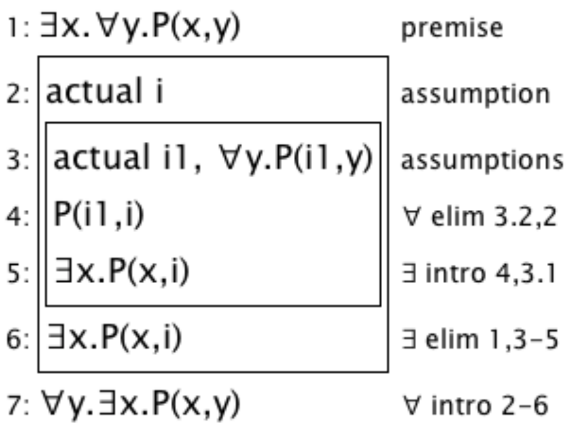
4.	$\forall y\, p(a, y)$		1, $\exists$ rule
5.	$\neg\exists x\, p(x, b)$	✓	3, $\exists$ rule
6.	$p(a, b)$		4, $\forall$ rule
7.	$\forall x\, \neg p(x, b)$		5, $\neg\exists$ rule
8.	$\neg p(a, b)$		7, $\forall$ rule
	X closes (6, 8)		

Example: Prove ∃x ∀y p(x, y) |– ∀y∃x p(x, y)

1.	∃x ∀y p(x, y)	✓	LHS
2.	¬∀y∃x p(x, y)	✓	RHS negated
3.	∃y¬∃x p(x, y)	✓	2, ¬∀ rule
4.	∀y p(a, y)		1, ∃ rule
5.	¬∃x p(x, b)	✓	3, ∃ rule
6.	p(a, b)		4, ∀ rule
7.	∀x ¬p(x, b)		5, ¬∃ rule
8.	¬p(a, b)		7, ∀ rule
	X closes (6, 8)		

Corresponding natural deduction proof



1:	∃x.∀y.P(x,y)	premise
2:	actual i	assumption
3:	actual i1, ∀y.P(i1,y)	assumptions
4:	P(i1,i)	∀ elim 3.2,2
5:	∃x.P(x,i)	∃ intro 4,3.1
6:	∃x.P(x,i)	∃ elim 1,3-5
7:	∀y.∃x.P(x,y)	∀ intro 2-6

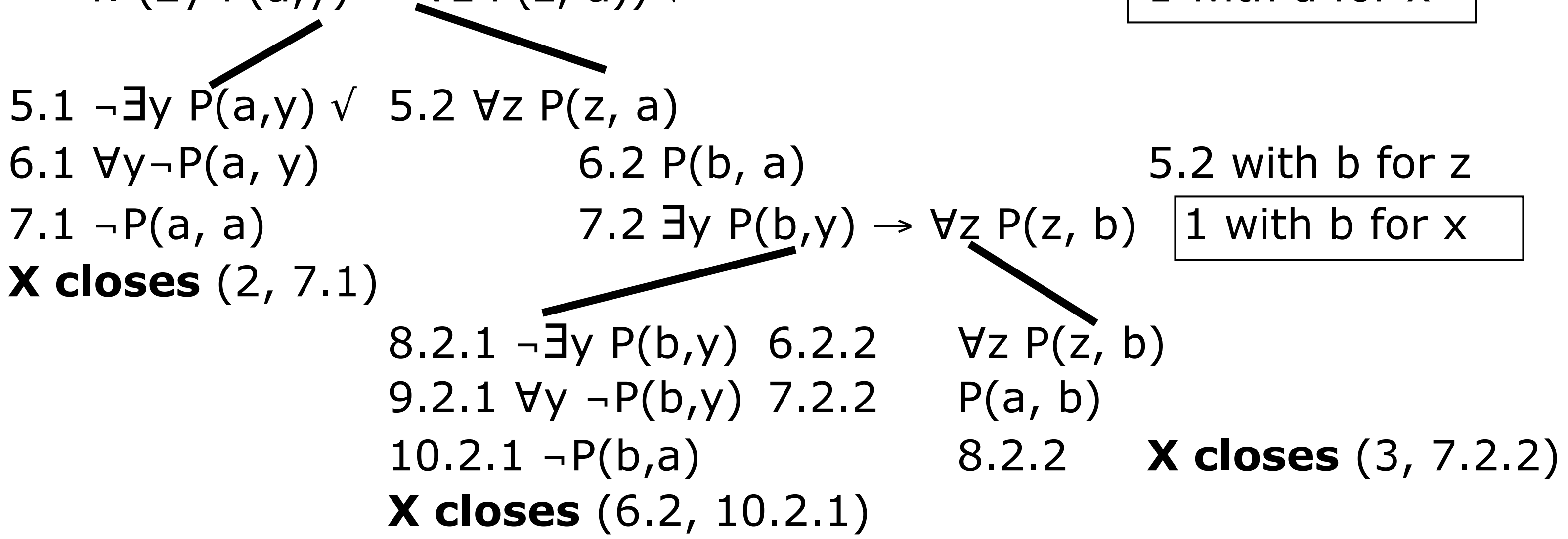
Example

(in which ∀ rule is used twice from the same line)

$\forall x\,(\exists y\,P(x,y) \rightarrow \forall z\,P(z,x)),\ P(a,a)\ \vdash P(a,b)$

This time we number for better clarity.

1. $\forall x\,(\exists y\,P(x,y) \rightarrow \forall z\,P(z,x))$ premise
2. $P(a,a)$ premise
3. $\neg P(a,b)$ negated conclusion
4. $(\exists y\,P(a,y) \rightarrow \forall z\,P(z,a))$ √ 1 with a for x



5.1 $\neg\exists y\,P(a,y)$ √ 5.2 $\forall z\,P(z,a)$

6.1 $\forall y \neg P(a,y)$ 6.2 $P(b,a)$ 5.2 with b for z

7.1 $\neg P(a,a)$ 7.2 $\exists y\,P(b,y) \rightarrow \forall z\,P(z,b)$ 1 with b for x

X closes (2, 7.1)

8.2.1 $\neg\exists y\,P(b,y)$ 6.2.2 $\forall z\,P(z,b)$

9.2.1 $\forall y\,\neg P(b,y)$ 7.2.2 $P(a,b)$

10.2.1 $\neg P(b,a)$ 8.2.2 **X closes** (3, 7.2.2)

X closes (6.2, 10.2.1)

Choice of Terms for ∀ rule

- Prefer terms constructed of constants introduced earlier [as they are more likely to produce closure].
- Introduce additional new constants as needed.
- Adding new constants implicitly invokes the **non-empty domain assumption**.

Example (already negated)

$$\neg(\forall x\ p(x) \rightarrow \exists x\ p(x))$$

Example

$\neg(\forall x\ p(x) \rightarrow \exists x\ p(x))$ ✓

$\forall x\ p(x)$

$\neg\exists x\ p(x)$

Example

$\neg(\forall x\ p(x) \rightarrow \exists x\ p(x))$ ✓

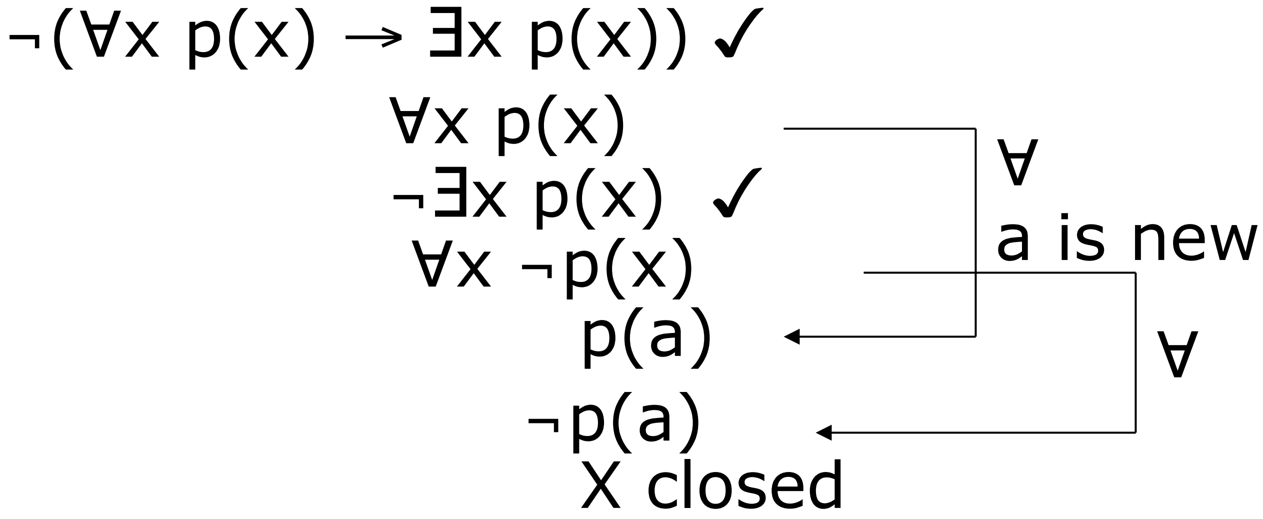
$\forall x\ p(x)$

$\neg\exists x\ p(x)$ ✓

$\forall x\ \neg p(x)$

To proceed further, we introduce a new constant ‘a’, then use the ∀x rule (twice).

Example



$\neg(\forall x\ p(x) \rightarrow \exists x\ p(x))$ ✓

$\forall x\ p(x)$

$\neg\exists x\ p(x)$ ✓

$\forall x\ \neg p(x)$

$p(a)$

$\neg p(a)$

X closed

The root formula is not satisfiable.

Thus $\forall x\ p(x) \rightarrow \exists x\ p(x)$ is valid

Closure depended on appropriate choice of term to substitute for x in $\forall x\ \neg p(x)$.

Termination

- Unlike the propositional case, the predicate version of tableaux **does not necessarily terminate**! This is because the ∀ rule can be used arbitrarily-many times.
- It can be shown, however, that **if** the root formula is **un**satisfiable, then **there exists** a closed tree for it.
- (If the root formula **is** satisfiable, the construction *might* not terminate.)

Example of Non-Termination

$$\forall x\, \exists y\, P(x, y) \vdash P(a, a)$$

1.	$\forall x\, \exists y\, P(x, y)$	premise
2.	$\neg P(a, a)$	conclusion negated
3.	$\exists y\, P(a, y)$ √	1, a for x
4.	$P(a, b)$	3, b for y
5.	$\exists y\, P(b, y)$ √	1, b for x
6.	$P(b, c)$	5, c for y
7.	$\exists y\, P(c, y)$ √	1, c for x
8.	$P(c, d)$	7, d for y
9.	$\exists y\, P(d, y)$ √	1, d for x
	. . .	. . .

Using the Tableau Method to
Find a **Counter-Example** in the Predicate Calculus

- A counter-example is a model for the negated formula.
- $\forall x\ (A(x) \vee B(x)) \rightarrow (\forall x\ A(x) \vee \forall x\ B(x))$
- This formula is not valid, so its negation should be satisfiable.

$$\neg(\forall x\ (A(x) \vee B(x)) \rightarrow (\forall x\ A(x) \vee \forall x\ B(x)))\ \checkmark$$

$$\forall x\ (A(x) \vee B(x))$$

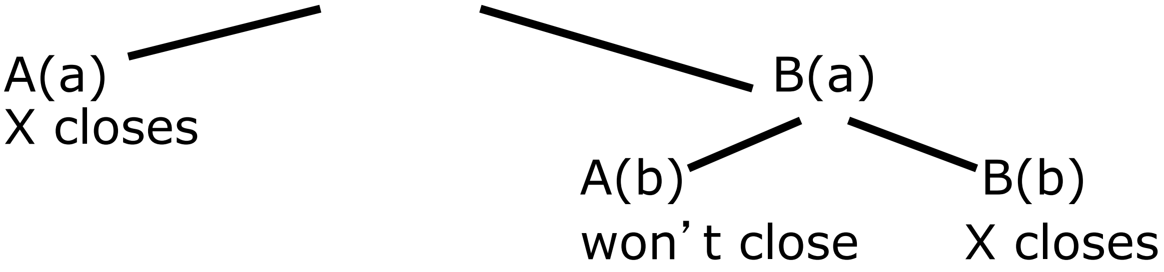
$$\neg(\forall x\ A(x) \vee \forall x\ B(x))\ \checkmark$$

$$\neg\forall x\ A(x)\ \checkmark$$
$$\neg\forall x\ B(x)\ \checkmark$$

$$\exists x\ \neg A(x)$$
$$\exists x\ \neg B(x)$$

Using the Tableau Method to Find a Model in the Predicate Calculus

$\forall x\ (A(x) \vee B(x))$
$\exists x\ \neg A(x)$ ✓
$\exists x\ \neg B(x)$ ✓
$\neg A(a)$
$\neg B(b)$
$A(a) \vee B(a)$ ✓
$A(b) \vee B(b)$ ✓



$A(a)$
X closes

$B(a)$

$A(b)$
won’t close

$B(b)$
X closes

Conclusion: **There is a model for the negation** with domain {a, b}, in which “¬A(a), A(b), B(a), and ¬B(b)” (translated into the appropriate interpretation notation).

Check for Counterexample

- A model that shows the negation is satisfiable is a counterexample for the validity of the original.

- $\forall x\ (A(x) \vee B(x)) \rightarrow (\forall x\ A(x) \vee \forall x\ B(x))$

- I: $\neg A(a)$, $A(b)$, $B(a)$, and $\neg B(b)$
- Domain = $\{a, b\}$

- $I[\forall x\ (A(x) \vee B(x))] = T$, but
  $I[\forall x\ A(x) \vee \forall x\ B(x)] = F$.

Handling Equality in Tableaux

- If an open path has a node $t_1 = t_2$, then for any unchecked node φ containing t_1, add on the path a formula in which t_1 is **replaced with** t_2 (and vice-versa), so long as appropriate rules for substitution are observed.

Example: Equality in Tableau

- P(a)∨P(b), ¬P(a) $\vdash$ ¬(a=b)
- Proof:

1. P(a)∨P(b)	premise
2. ¬P(a)	premise
3. ¬¬(a=b) ✓	negated conclusion
4. a=b	3, ¬¬
5. ¬P(b)	2, 4, = rule

P(a) P(b) 1

X closes X closes

Example: Equality in Tableau

- $a = b \vdash P(a, b) \rightarrow P(b, a)$
- Proof:

1. $a = b$	premise
2. $\neg(P(a, b) \rightarrow P(b, a))$ ✓	negated conclusion
3. $P(a, b)$	2
4. $\neg P(b, a)$	2
5. $P(a, a)$	3, 1, = rule
6. $\neg P(a, a)$	4, 1, = rule
X closes(5, 6)	

Sequent Calculus for Predicates

- As with the tableau method, the propositional rules for Sequent Calculus will be augmented with four **new rules for quantifiers**.

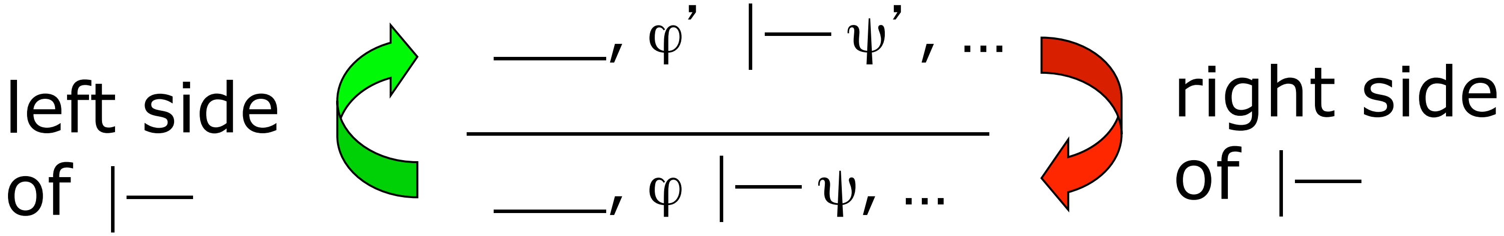
- As before, the Sequent Calculus rules have a correspondence with the tableau proof rules.

- Whereas the tableau shows negation explicitly, in Sequent Calculus it may be implicit, depending on which side of the turnstile a formula appears.

- **Negated formulas in tableaux** generally correspond to **formulas on the right** of the turnstile in Sequent Calculus.

Mind Set for Remembering Sequent Calculus

- Think of the sequent(s) above the line as being **sufficient** to prove the one below.
- Think of the "information flow" as shown in the diagram.



left side of |—

$$\frac{\underline{\qquad}, \varphi' \vdash \psi', \ldots}{\underline{\qquad}, \varphi \vdash \psi, \ldots}$$

right side of |—

∀L rule

$$\frac{\underline{\qquad}, \varphi[t/x] \;\vdash \ldots}{\underline{\qquad}, \forall x\, \varphi \;\vdash \ldots}\ \forall L$$

where t is any term.

This rule parallels the **∀-Elimination** rule of natural deduction.

It says that … can be proved from ____, $\forall x\, \varphi$ provided that it can be proved from ____, $\varphi[t/x]$.

∃L rule

$$\dfrac{\_\_\_,\ \varphi[x_0/x]\ \vdash\ \ldots}{\_\_\_,\ \exists x\ \varphi\ \vdash\ \ldots}\ \exists L$$

where x_0 is a fresh variable not occurring in ___ or … .

This rule parallels the **∃-Elimination** rule of natural deduction.

It says that … can be proved from ___, $\exists x \varphi$ provided that it can be proved from ___, $\varphi[x_0/x]$.

∀R rule

$$\frac{\_\_\_\_ \mid\!\!-\, \varphi[x_0/x], \ldots}{\_\_\_\_ \mid\!\!-\, \forall x\ \varphi, \ldots} \mid\!\!-\, \forall R$$

where x_0 is a fresh variable not occurring in ____ or … .

This rule parallels the **∀-Introduction** rule of natural deduction.

It states that to prove $\forall x \varphi$ it suffices to prove $\varphi[x_0/x]$ where x_0 is an an arbitrary fresh variable.

∃R rule

$$\frac{\_\_\_\_ \ \vdash \varphi[t/x], \ldots}{\_\_\_\_ \ \vdash \exists x\ \varphi, \ldots} \ \vdash \exists R$$

where t is an term free for x in φ.

This rule parallels the **∃-Introduction rule** of natural deduction.

It states that to prove ∃xφ it suffices to prove φ[t/x] where t is any term.

Sequent Calculus Quantifier Rule Summary

	Left	Right
∃	$\dfrac{\_\_\_\_,\ \varphi[x_0/x]\ \vdash \ldots}{\_\_\_\_,\ \exists x\ \varphi\ \vdash \ldots}\ \exists L$	$\dfrac{\_\_\_\ \vdash \varphi[t/x],\ \ldots}{\_\_\_\ \vdash \exists x\ \varphi,\ \ldots}\ \vdash \exists R$
∀	$\dfrac{\_\_\_\_,\ \varphi[t/x]\ \vdash \ldots}{\_\_\_\_,\ \forall x\ \varphi\ \vdash \ldots}\ \forall L$	$\dfrac{\_\_\_\ \vdash \varphi[x_0/x],\ \ldots}{\_\_\_\ \vdash \forall x\ \varphi,\ \ldots}\ \vdash \forall R$

Contraction Rule (Occasionally Needed)

- Going from **below to above**, any formula can be **copied** to the **same** side (i.e. left-to-left, right-to-right).
- Reason: Adding a copy of a premise or conclusion doesn’t change validity.
- This type of rule is known as a “**structural**” rule.

Structural Rules, when elements are viewed as lists rather than multisets.

http://en.wikipedia.org/wiki/Sequent_calculus

Left structural rules:

Right structural rules:

W = “Weakening”

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ (WL)$$

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\ (WR)$$

C = “Contraction”

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\ (CL)$$

$$\frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\ (CR)$$

P = “Permutation”

$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}\ (PL)$$

$$\frac{\Gamma \vdash \Delta_1, A, B, \Delta_2}{\Gamma \vdash \Delta_1, B, A, \Delta_2}\ (PR)$$

Sequent Calculus Example

$\exists x \, \forall y \; p(x, y) \mid\!- \forall y \exists x \; p(x, y)$

Sequent Calculus Example

$$\frac{\_\_\_,\ \varphi[x_0/x]\ |\!-\ \ldots}{\_\_\_,\ \exists x\ \varphi\ |\!-\ \ldots}\ \exists L$$

$$a/x \qquad \frac{\forall y\ p(a, y)\ |\!-\ \forall y \exists x\ p(x, y)}{\exists x\ \forall y\ p(x, y)\ |\!-\ \forall y \exists x\ p(x, y)}\quad \exists L \quad a/x$$

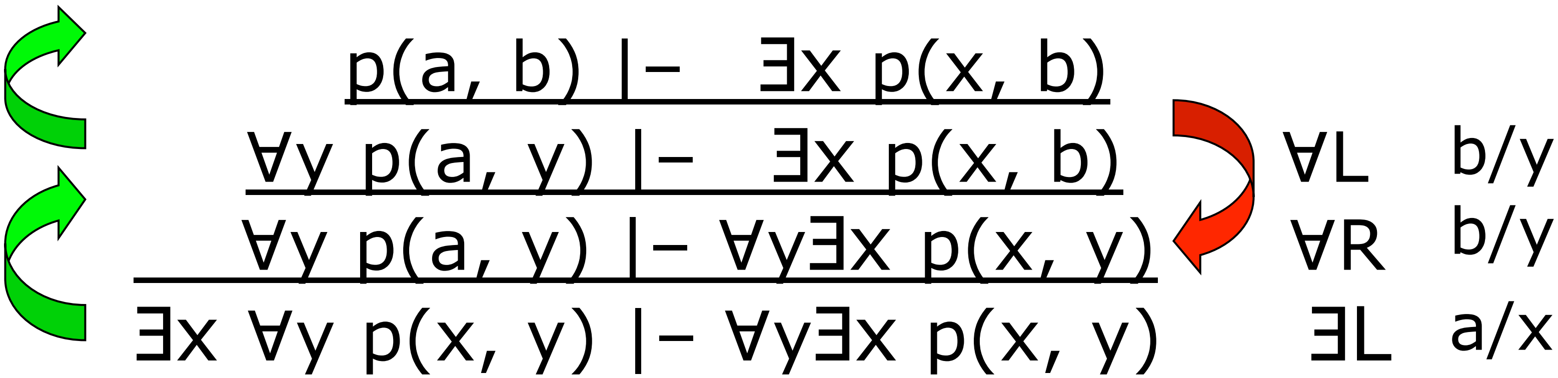
Sequent Calculus Example

$$\frac{\_\_\_\_ \;|\!- \varphi[x_0/x], \ldots}{\_\_\_\_ \;|\!- \forall x\ \varphi, \ldots}\ |\!- \forall R$$

$$\frac{\dfrac{\forall y\ p(a, y)\ |\!-\ \ \exists x\ p(x, b)}{\forall y\ p(a, y)\ |\!-\ \forall y \exists x\ p(x, y)}\ \forall R\quad b/y}{\exists x\ \forall y\ p(x, y)\ |\!-\ \forall y \exists x\ p(x, y)}\ \exists L\quad a/x$$

Sequent Calculus Example

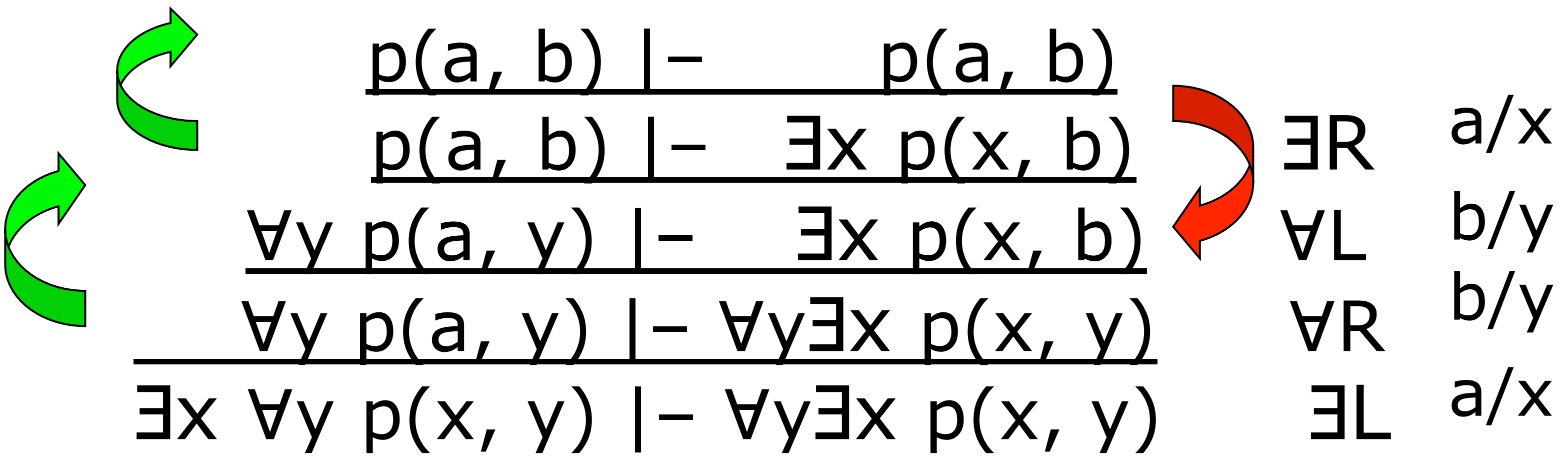
$$\frac{\_\_\_,\ \varphi[t/x]\ \mid\!\!—\ \dots}{\_\_\_,\ \forall x\ \varphi\ \mid\!\!—\ \dots}\ \forall L$$



$$\begin{array}{cll}
\underline{p(a, b)\ \mid\!- \quad \exists x\ p(x, b)} & & \\
\underline{\forall y\ p(a, y)\ \mid\!- \quad \exists x\ p(x, b)} & \forall L & b/y \\
\underline{\forall y\ p(a, y)\ \mid\!-\ \forall y \exists x\ p(x, y)} & \forall R & b/y \\
\exists x\ \forall y\ p(x, y)\ \mid\!-\ \forall y \exists x\ p(x, y) & \exists L & a/x
\end{array}$$

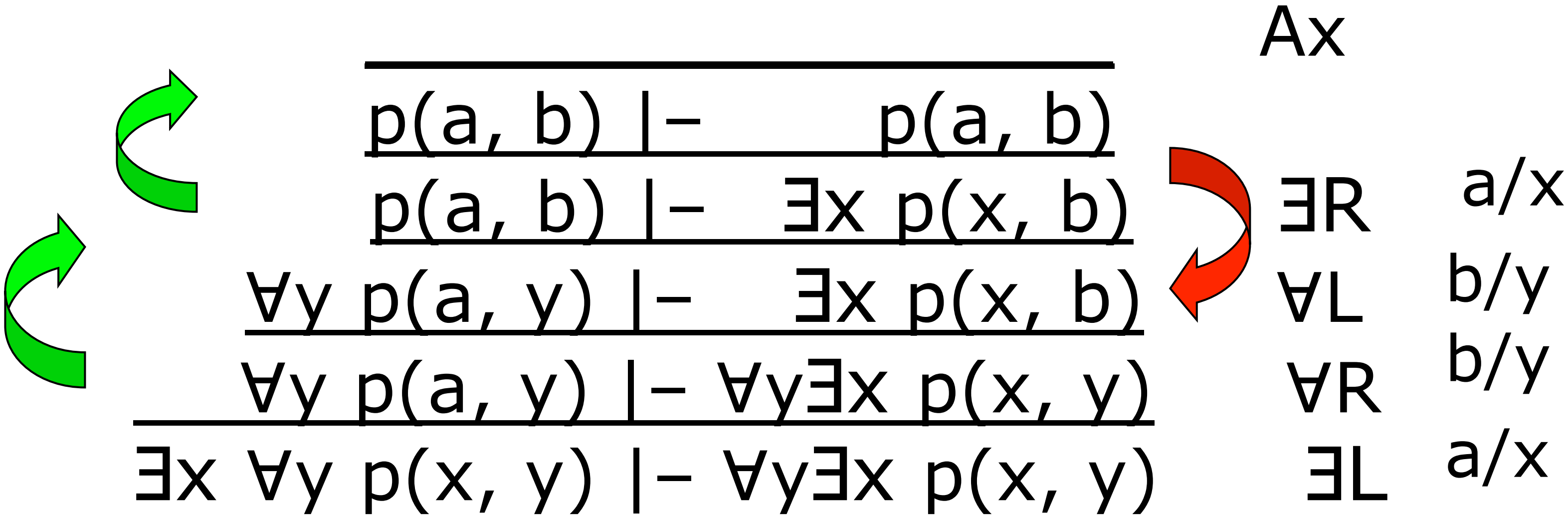
Sequent Calculus Example

$$\dfrac{\_\_\_ \ \vdash \varphi[t/x], \ldots}{\_\_\_ \ \vdash \exists x\ \varphi, \ldots} \ \vdash \exists R$$



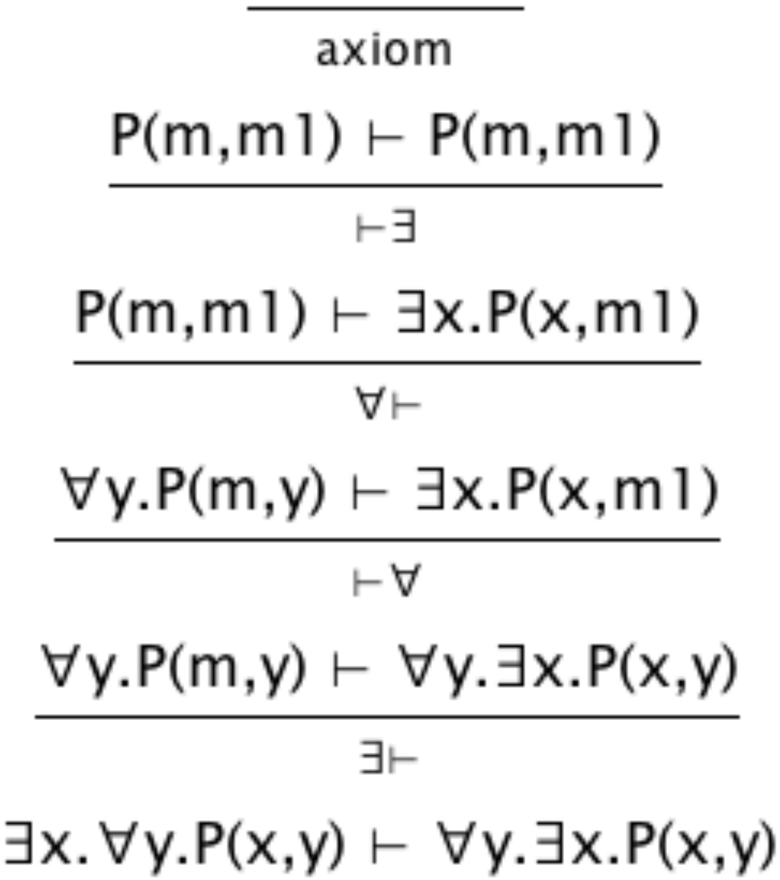
$$\begin{array}{lll}
\underline{p(a, b)\ |\!-\ \quad p(a, b)} & & \\
\underline{p(a, b)\ |\!-\ \exists x\ p(x, b)} & \exists R & a/x \\
\underline{\forall y\ p(a, y)\ |\!-\ \exists x\ p(x, b)} & \forall L & b/y \\
\underline{\forall y\ p(a, y)\ |\!-\ \forall y \exists x\ p(x, y)} & \forall R & b/y \\
\exists x\ \forall y\ p(x, y)\ |\!-\ \forall y \exists x\ p(x, y) & \exists L & a/x
\end{array}$$

Sequent Calculus Example



$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p(a, b) \mid\!- \quad p(a, b)}\;Ax}{p(a, b) \mid\!- \quad \exists x\ p(x, b)}\;\exists R \quad a/x}{\forall y\ p(a, y) \mid\!- \quad \exists x\ p(x, b)}\;\forall L \quad b/y}{\forall y\ p(a, y) \mid\!- \forall y \exists x\ p(x, y)}\;\forall R \quad b/y}{\exists x\ \forall y\ p(x, y) \mid\!- \forall y \exists x\ p(x, y)}\;\exists L \quad a/x$$

Same Example, in JAPE



$$\begin{array}{c}
\hline
\text{axiom} \\
P(m,m1) \vdash P(m,m1) \\
\hline
\vdash\exists \\
P(m,m1) \vdash \exists x.P(x,m1) \\
\hline
\forall\vdash \\
\forall y.P(m,y) \vdash \exists x.P(x,m1) \\
\hline
\vdash\forall \\
\forall y.P(m,y) \vdash \forall y.\exists x.P(x,y) \\
\hline
\exists\vdash \\
\exists x.\forall y.P(x,y) \vdash \forall y.\exists x.P(x,y)
\end{array}$$

Variant, Using Explicit Substitutions

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P[x,y\backslash m,m1] \vdash P[y,x\backslash m1,m]}\,\text{axiom}}{P[x,y\backslash m,m1] \vdash \exists x.P[y\backslash m1]}\,{\vdash}\exists}{\forall y.P[x\backslash m] \vdash \exists x.P[y\backslash m1]}\,\forall{\vdash}}{\forall y.P[x\backslash m] \vdash \forall y.\exists x.P}\,{\vdash}\forall}{\exists x.\forall y.P \vdash \forall y.\exists x.P}\,\exists{\vdash}$$

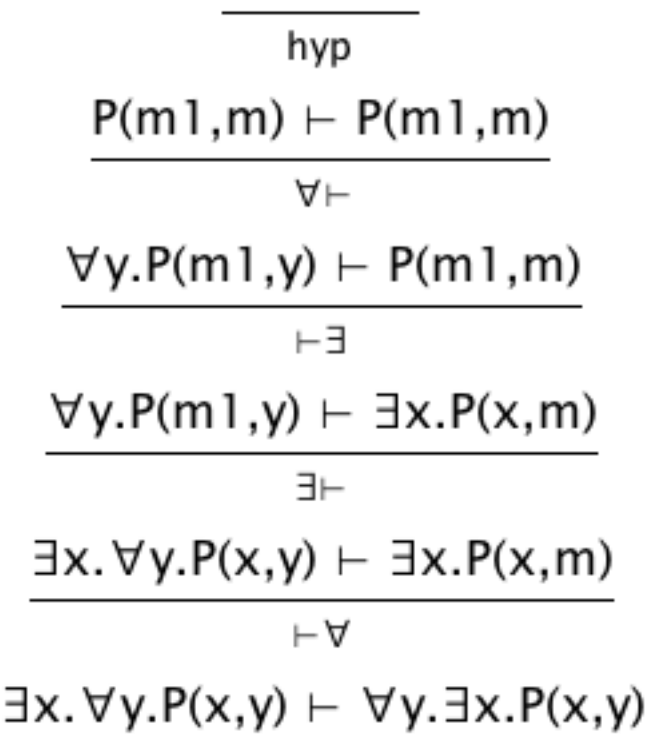
x\m means: m substituted for x

x,y\m,m1 means: m substituted for x, m1 for y

Single-Consequent Sequent Calculus = Intuitionistic Logic

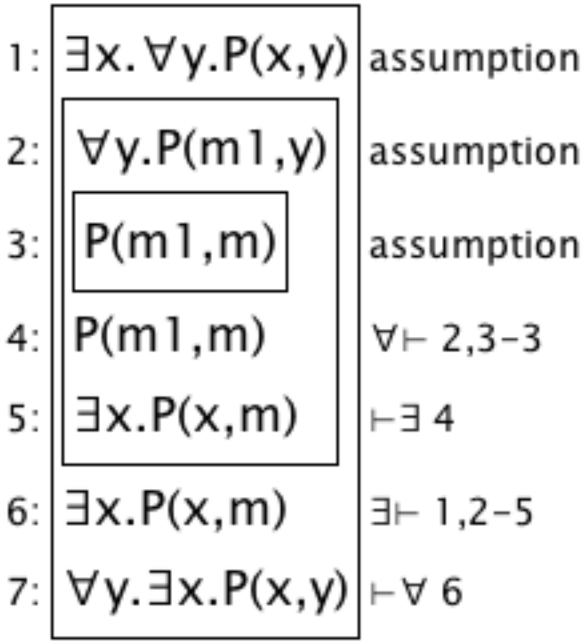
$$\begin{array}{c}
\hline
\text{hyp} \\
P(m1,m) \vdash P(m1,m) \\
\hline
\forall\vdash \\
\forall y.P(m1,y) \vdash P(m1,m) \\
\hline
\vdash\exists \\
\forall y.P(m1,y) \vdash \exists x.P(x,m) \\
\hline
\exists\vdash \\
\exists x.\forall y.P(x,y) \vdash \exists x.P(x,m) \\
\hline
\vdash\forall \\
\exists x.\forall y.P(x,y) \vdash \forall y.\exists x.P(x,y)
\end{array}$$

Single Consequent SC JAPE
Automates conversion to Natural Deduction



$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(m1,m) \vdash P(m1,m)}\,\text{hyp}}{\forall y.P(m1,y) \vdash P(m1,m)}\,\forall\vdash}{\forall y.P(m1,y) \vdash \exists x.P(x,m)}\,\vdash\exists}{\exists x.\forall y.P(x,y) \vdash \exists x.P(x,m)}\,\exists\vdash}{\exists x.\forall y.P(x,y) \vdash \forall y.\exists x.P(x,y)}\,\vdash\forall$$

Select
Box Display

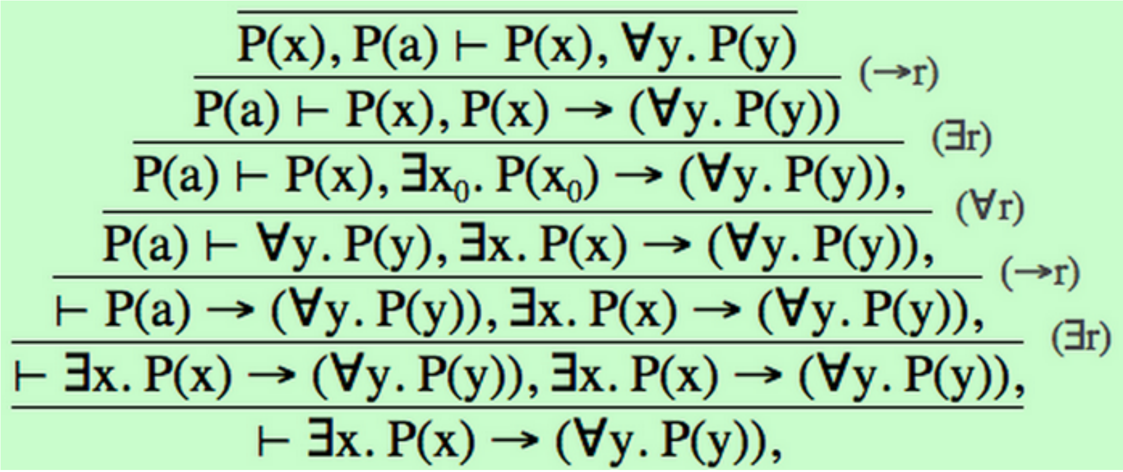


1:	$\exists x.\forall y.P(x,y)$	assumption
2:	$\forall y.P(m1,y)$	assumption
3:	$P(m1,m)$	assumption
4:	$P(m1,m)$	$\forall\vdash$ 2,3-3
5:	$\exists x.P(x,m)$	$\vdash\exists$ 4
6:	$\exists x.P(x,m)$	$\exists\vdash$ 1,2-5
7:	$\forall y.\exists x.P(x,y)$	$\vdash\forall$ 6

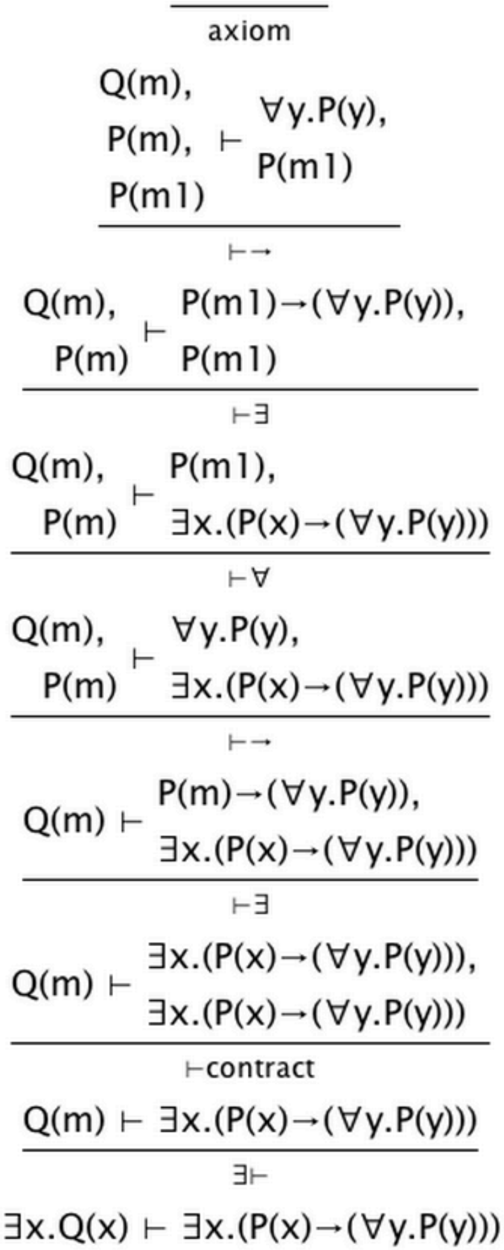
Interactive SC Tutorial

- http://logitext.mit.edu/logitext.fcgi/tutorial
- There is a small problem with variable renaming, but hopefully it will get fixed.

Logitext vs. JAPE MCS



$$\begin{array}{cl}
\hline
P(x), P(a) \vdash P(x), \forall y.\, P(y) & \\
\hline
P(a) \vdash P(x), P(x) \rightarrow (\forall y.\, P(y)) & (\rightarrow r) \\
\hline
P(a) \vdash P(x), \exists x_0.\, P(x_0) \rightarrow (\forall y.\, P(y)), & (\exists r) \\
\hline
P(a) \vdash \forall y.\, P(y), \exists x.\, P(x) \rightarrow (\forall y.\, P(y)), & (\forall r) \\
\hline
\vdash P(a) \rightarrow (\forall y.\, P(y)), \exists x.\, P(x) \rightarrow (\forall y.\, P(y)), & (\rightarrow r) \\
\hline
\vdash \exists x.\, P(x) \rightarrow (\forall y.\, P(y)), \exists x.\, P(x) \rightarrow (\forall y.\, P(y)), & (\exists r) \\
\hline
\vdash \exists x.\, P(x) \rightarrow (\forall y.\, P(y)), &
\end{array}$$



$$\begin{array}{c}
\hline
\text{axiom} \\
Q(m), P(m), P(m1) \vdash \forall y.P(y), P(m1) \\
\hline
\vdash\rightarrow \\
Q(m), P(m) \vdash P(m1)\rightarrow(\forall y.P(y)), P(m1) \\
\hline
\vdash\exists \\
Q(m), P(m) \vdash P(m1), \exists x.(P(x)\rightarrow(\forall y.P(y))) \\
\hline
\vdash\forall \\
Q(m), P(m) \vdash \forall y.P(y), \exists x.(P(x)\rightarrow(\forall y.P(y))) \\
\hline
\vdash\rightarrow \\
Q(m) \vdash P(m)\rightarrow(\forall y.P(y)), \exists x.(P(x)\rightarrow(\forall y.P(y))) \\
\hline
\vdash\exists \\
Q(m) \vdash \exists x.(P(x)\rightarrow(\forall y.P(y))), \exists x.(P(x)\rightarrow(\forall y.P(y))) \\
\hline
\vdash\text{contract} \\
Q(m) \vdash \exists x.(P(x)\rightarrow(\forall y.P(y))) \\
\hline
\exists\vdash \\
\exists x.Q(x) \vdash \exists x.(P(x)\rightarrow(\forall y.P(y)))
\end{array}$$

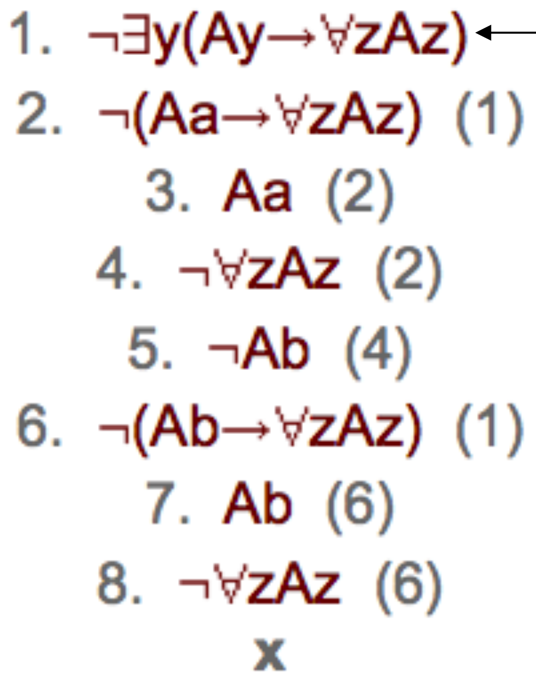
Q is an artifact to assume a non-empty universe.

Rule Correspondence: Tableau-SC

Tableau Rule (going downward)	Sequent Calculus Rule (going backward/upward)
¬∃ replace with ∀¬	∃R specializes term on right
¬∀ replace with ∃¬	∀R specializes term on right to fresh var
∃ introduces constant	∃L specializes term on left to fresh var
∀ uses term	∀L specializes term on left
closing path	axiom
reusing term	contraction rule

Automated Tableau Proof Example: |— ∃y (A(y) → ∀z A(z))

http://www.umsu.de/logik/trees/



1. ¬∃y(Ay→∀zAz)
2. ¬(Aa→∀zAz) (1)
3. Aa (2)
4. ¬∀zAz (2)
5. ¬Ab (4)
6. ¬(Ab→∀zAz) (1)
7. Ab (6)
8. ¬∀zAz (6)

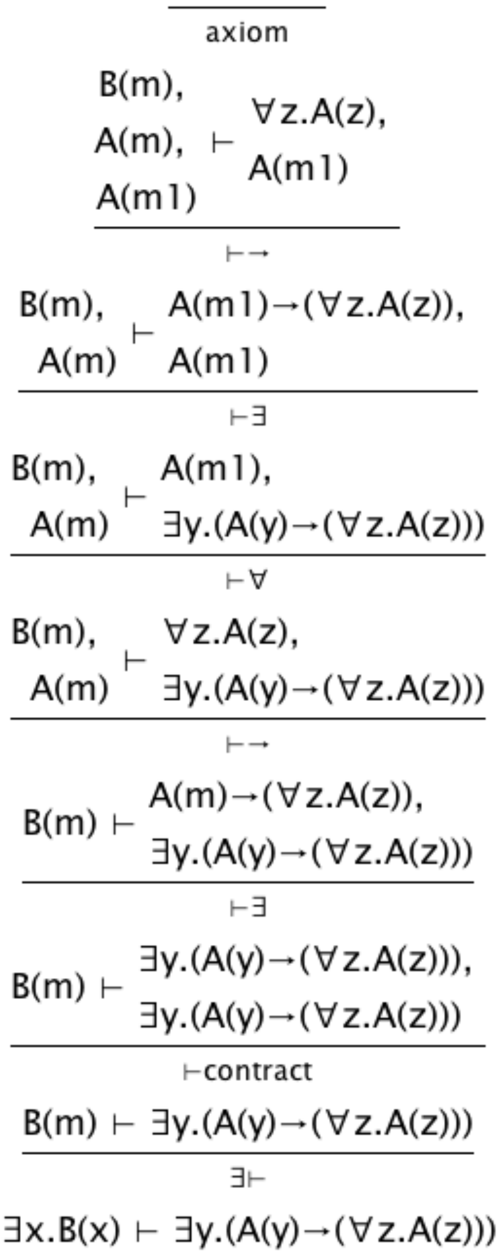
x

Stacking ∀
is **implicit** in
this tool.

Two uses of (1)

corresponds to
contraction rule

JAPE Sequent Calculus Proof



$$\begin{array}{c}
\hline
\text{axiom} \\
B(m), A(m), A(m1) \vdash \forall z.A(z), A(m1) \\
\hline
\vdash\rightarrow \\
B(m), A(m) \vdash A(m1)\rightarrow(\forall z.A(z)), A(m1) \\
\hline
\vdash\exists \\
B(m), A(m) \vdash A(m1), \exists y.(A(y)\rightarrow(\forall z.A(z))) \\
\hline
\vdash\forall \\
B(m), A(m) \vdash \forall z.A(z), \exists y.(A(y)\rightarrow(\forall z.A(z))) \\
\hline
\vdash\rightarrow \\
B(m) \vdash A(m)\rightarrow(\forall z.A(z)), \exists y.(A(y)\rightarrow(\forall z.A(z))) \\
\hline
\vdash\exists \\
B(m) \vdash \exists y.(A(y)\rightarrow(\forall z.A(z))), \exists y.(A(y)\rightarrow(\forall z.A(z))) \\
\hline
\vdash\text{contract} \\
B(m) \vdash \exists y.(A(y)\rightarrow(\forall z.A(z))) \\
\hline
\exists\vdash \\
\exists x.B(x) \vdash \exists y.(A(y)\rightarrow(\forall z.A(z)))
\end{array}$$

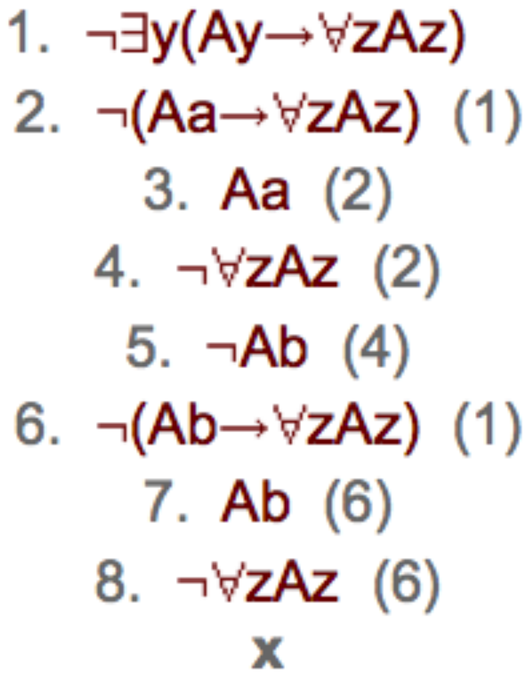
B is an artifact to mean something exists.
It plays no other role.

Contraction Rule used to get two instances

JAPE Sequent Calculus Proof

B just means something exists

Tableau Proof
for Comparison

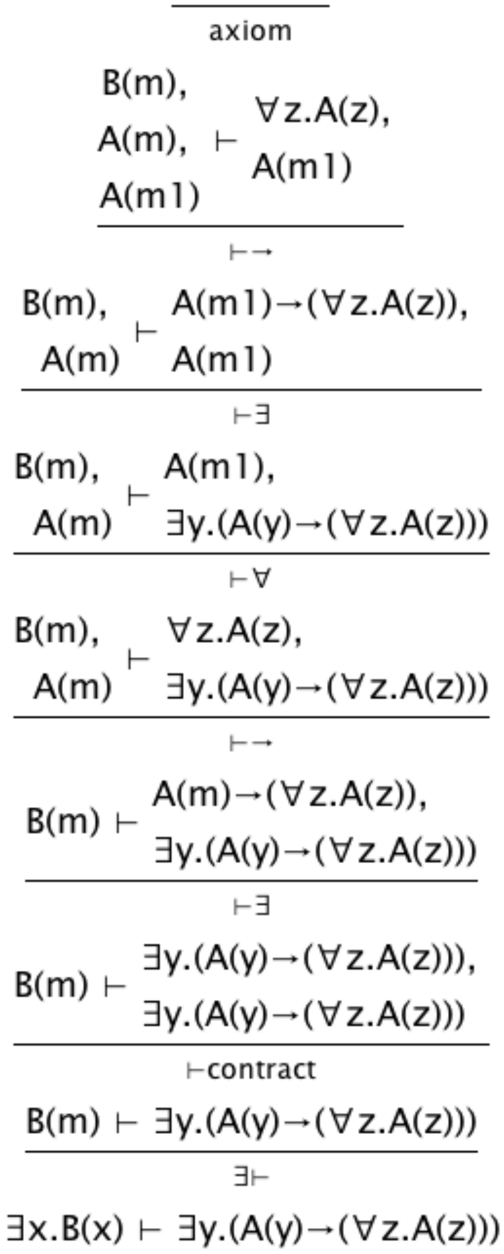


1. $\neg\exists y(Ay \rightarrow \forall zAz)$
2. $\neg(Aa \rightarrow \forall zAz)$ (1)
3. Aa (2)
4. $\neg\forall zAz$ (2)
5. $\neg Ab$ (4)
6. $\neg(Ab \rightarrow \forall zAz)$ (1)
7. Ab (6)
8. $\neg\forall zAz$ (6)

x

Inverted Tableau

8. $\neg\forall zA(z)$
7. $A(b)$
6. $\neg(A(b) \rightarrow \forall zA(z))$
5. $\neg A(b)$
4. $\neg\forall zA(z)$
3. $A(a)$
2. $\neg(A(a) \rightarrow \forall zA(z))$
1. $\neg\exists y(A(y) \rightarrow \forall zA(z))$



axiom

$B(m), A(m), A(m1) \vdash \forall z.A(z), A(m1)$

$\vdash\rightarrow$

$B(m), A(m) \vdash A(m1)\rightarrow(\forall z.A(z)), A(m1)$

$\vdash\exists$

$B(m), A(m) \vdash A(m1), \exists y.(A(y)\rightarrow(\forall z.A(z)))$

$\vdash\forall$

$B(m), A(m) \vdash \forall z.A(z), \exists y.(A(y)\rightarrow(\forall z.A(z)))$

$\vdash\rightarrow$

$B(m) \vdash A(m)\rightarrow(\forall z.A(z)), \exists y.(A(y)\rightarrow(\forall z.A(z)))$

$\vdash\exists$

$B(m) \vdash \exists y.(A(y)\rightarrow(\forall z.A(z))), \exists y.(A(y)\rightarrow(\forall z.A(z)))$

$\vdash$contract

$B(m) \vdash \exists y.(A(y)\rightarrow(\forall z.A(z)))$

$\exists\vdash$

$\exists x.B(x) \vdash \exists y.(A(y)\rightarrow(\forall z.A(z)))$

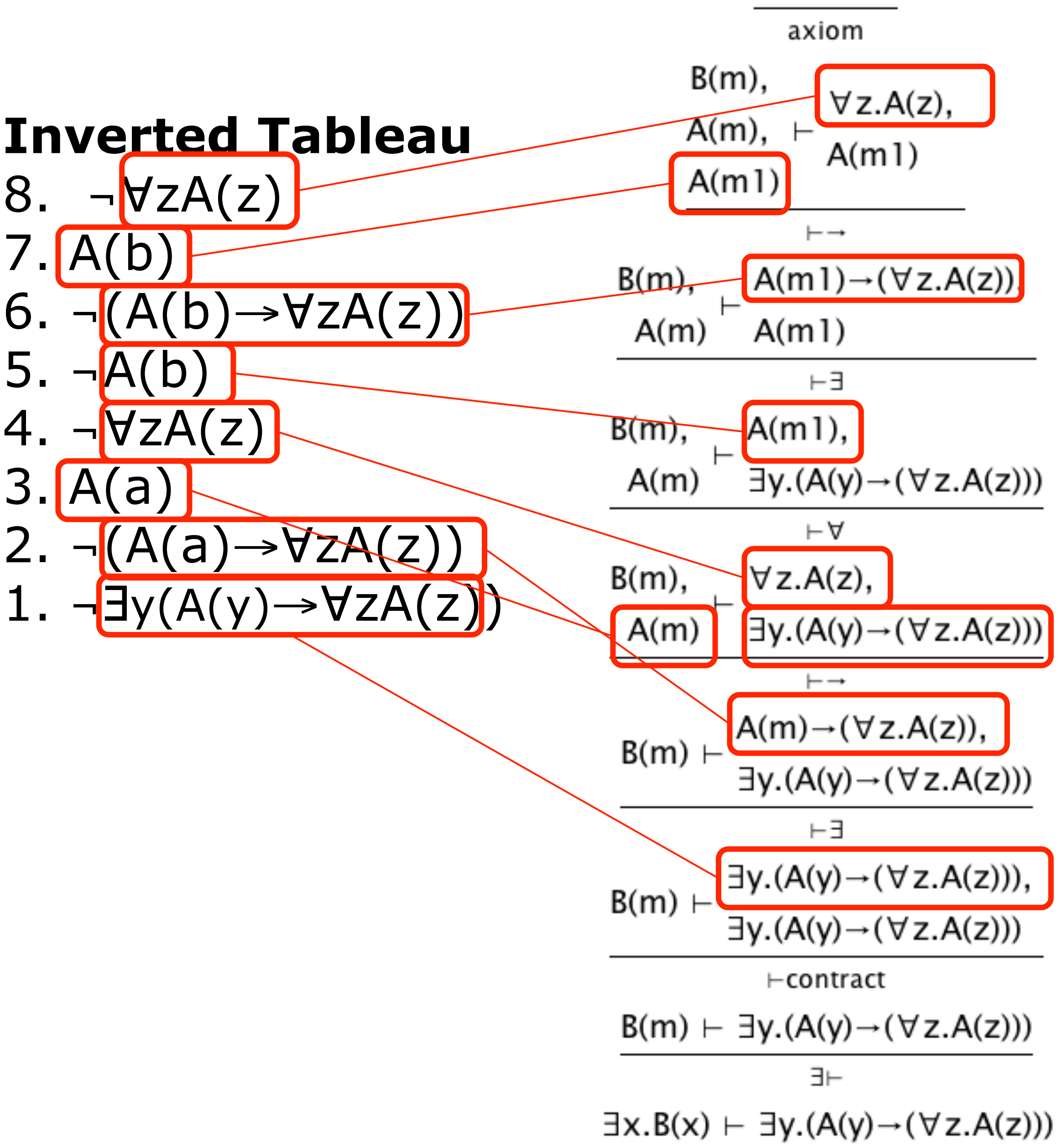
Connecting the Dots

B just means something exists

Tableau Proof
for Comparison

1. ¬∃y(Ay→∀zAz)
2. ¬(Aa→∀zAz) (1)
3. Aa (2)
4. ¬∀zAz (2)
5. ¬Ab (4)
6. ¬(Ab→∀zAz) (1)
7. Ab (6)
8. ¬∀zAz (6)

x



Inverted Tableau

8. ¬∀zA(z)
7. A(b)
6. ¬(A(b)→∀zA(z))
5. ¬A(b)
4. ¬∀zA(z)
3. A(a)
2. ¬(A(a)→∀zA(z))
1. ¬∃y(A(y)→∀zA(z))

axiom

B(m), A(m), A(m1) ⊢ ∀z.A(z), A(m1)

⊢→

B(m), A(m) ⊢ A(m1)→(∀z.A(z)), A(m1)

⊢∃

B(m), A(m) ⊢ A(m1), ∃y.(A(y)→(∀z.A(z)))

⊢∀

B(m), A(m) ⊢ ∀z.A(z), ∃y.(A(y)→(∀z.A(z)))

⊢→

B(m) ⊢ A(m)→(∀z.A(z)), ∃y.(A(y)→(∀z.A(z)))

⊢∃

B(m) ⊢ ∃y.(A(y)→(∀z.A(z))), ∃y.(A(y)→(∀z.A(z)))

⊢contract

B(m) ⊢ ∃y.(A(y)→(∀z.A(z)))

∃⊢

∃x.B(x) ⊢ ∃y.(A(y)→(∀z.A(z)))

Tableau/Sequent Calculus Correspondence

Inverted Tableau	**Sequent Left**	**Sequent Right (Implicitly Negated)**
8. ¬∀zA(z)		∀zA(z)
7. A(b)	A(b)	
6. ¬(A(b)→∀zA(z))		A(b)→∀zA(z)
5. ¬A(b)		A(b)
4. ¬∀zA(z)		∀zA(z)
3. A(a)	A(a)	
2. ¬(A(a)→∀zA(z))		A(a)→∀zA(z)
1. ¬∃y(A(y)→∀zA(z))		∃y(A(y)→∀zA(z))