Semantics, Soundness, and Completeness for Propositional Natural Deduction

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Proof vs. Truth

- So far, we have seen a method (natural deduction) for **proof** of formulas.

- It would be nice if we had an **independent** definition of **truth** of those formulas so that we could ascertain whether

  - Our proofs are proving only *true* statements. (**soundness**)

  - There is nothing lacking in our proof system. (**completeness**)

Giving Formulas a Meaning

- An **valuation** is a function ν (Greek "nu") that associates a value in {T, F} to every proposition symbol, with the requirement that, for the special symbols T and $\perp$ ("top" and "bottom"):
  - $\nu(\top) = T$
  - $\nu(\perp) = F$
- In the range of ν, T is intended to represent "true" and F "false".
- A valuation is variously called an *assignment*, *interpretation*, or (in CS) *environment* (depending on the author or text).

Induced Values for Formulas

- An **valuation** ν ***induces*** a value $\nu(\varphi)$ in {T, F} in any formula φ, inductively as follows:

Formula φ	**Value** $\nu(\varphi)$	
single proposition symbol p	ν(p) as given by the valuation ν	
$\neg F$	$h_{\neg}(\nu(F))$	where $h_{\neg}(F) = T$ and $h_{\neg}(T) = F$
$F \wedge G$	$h_{\wedge}(\nu(F), \nu(G))$	where $h_{\wedge}(x, y) = T$ iff $x = T$ **and** $y = T$
$F \vee G$	$h_{\vee}(\nu(F), \nu(G))$	where $h_{\vee}(x, y) = T$ iff $x = T$ **or** $y = T$
$F \rightarrow G$	$h_{\rightarrow}(\nu(F), \nu(G))$	where $h_{\rightarrow}(x, y) = T$ iff $x = F$ or $y = T$
$F \leftrightarrow G$	$h_{\leftrightarrow}(\nu(F), \nu(G))$	where $h_{\leftrightarrow}(x, y) = T$ iff $x = y$

Truth Function Summary

x	y	$h_\neg(y)$	$h_\wedge(x, y)$	$h_\vee(x, y)$	$h_\rightarrow(x, y)$	$h_\leftrightarrow(x, y)$
F	F	T	F	F	T	T
F	T	F	F	T	T	F
T	F		F	T	F	F
T	T		T	T	T	T

Example of Induced Value

- Formula: $p \vee q \rightarrow \neg p \wedge q$
- Valuation: $\nu(p) = F$, $\nu(q) = T$, …
- Induced Value: $\nu(p \vee q \rightarrow \neg p \wedge q)$

$$
\begin{aligned}
&= h_{\rightarrow}(\nu(p \vee q), \quad \nu(\neg p \wedge q)) \\
&= h_{\rightarrow}(h_{\vee}(\nu(p), \nu(q)), \quad h_{\wedge}(h_{\neg}(\nu(p)), \nu(q)) \\
&= h_{\rightarrow}(h_{\vee}(F, T), \quad h_{\wedge}(h_{\neg}(F), T) \\
&= h_{\rightarrow}(T, \quad h_{\wedge}(T, T)) \\
&= h_{\rightarrow}(T, T) \\
&= T
\end{aligned}
$$

Another Example of Induced Value

- Formula: $p \vee q \rightarrow \neg p \wedge q$

- Valuation: $\nu(p) = T$, $\nu(q) = F$, …

- Induced Value: $\nu(p \vee q \rightarrow \neg p \wedge q)$

$$
\begin{aligned}
&= h_{\rightarrow}(\nu(p \vee q), && \nu(\neg p \wedge q)) \\
&= h_{\rightarrow}(h_{\vee}(\nu(p), \nu(q)), && h_{\wedge}(h_{\neg}(\nu(p)), \nu(q)) \\
&= h_{\rightarrow}(h_{\vee}(T, F), && h_{\wedge}(h_{\neg}(T), F) \\
&= h_{\rightarrow}(T, && h_{\wedge}(F, F)) \\
&= h_{\rightarrow}(T, F) \\
&= F
\end{aligned}
$$

Language Interpreter

- Note that the determination of the induced value of an expression given a valuation is essentially defining an **interpreter** for the language.

- The valuation would typically be called an “environment” in that context.

Satisfaction Definition

- A valuation ν **satisfies** a formula φ iff the induced value $\nu(\varphi) = T$.

- A formula is **satisfiable** iff there is some valuation that satisfies it. Otherwise it is **unsatisfiable**.

- Examples:
  - $p \to \neg p$ is satisfiable (for what valuation?)
  - $p \wedge \neg p$ is unsatisfiable

Semantic Entailment: Double Turnstile

- Let $\varphi_1, \ldots, \varphi_n, \psi$ be formulas.
- The meaning of

$$\varphi_1, \ldots, \varphi_n \models \psi$$

is:

For every valuation ν such that

ν satisfies each of $\varphi_1, \ldots, \varphi_n$, (*)

ν also satisfies ψ. (§)

Example of Entailment $\models$

- Determine whether $p \vee q, \neg q \vee r \models p \vee r$
- We need to look at at most 8 valuations ν, one for each possible value of $\nu(p)$, $\nu(q)$, $\nu(r)$.

$\nu(p)$	$\nu(q)$	$\nu(r)$	$\nu(p \vee q)$	$\nu(\neg q \vee r)$	* holds (LHS)	§ holds (RHS)	$\nu(p \vee r)$
F	F	F	F	T			F
F	F	T	F	T		√	T
F	T	F	T	F			F
F	T	T	T	T	√	√	T
T	F	F	T	T	√	√	T
T	F	T	T	T	√	√	T
T	T	F	T	F		√	T
T	T	T	T	T	√	√	T

Example of $\models$

- Determine whether or not $p \vee q, \neg q \vee r \models p \vee r$

- **Alternatively**, we could *reason* as follows:
  - $\nu(q)$ = F or T.
  - If $\nu(q)$ = F, then * holds iff $\nu(p)$ = T,
    and in that case $\nu(p \vee r)$ = T, i.e. § holds.
  - If $\nu(q)$ = T, then * holds iff $\nu(r)$ = T,
    and in that case $\nu(p \vee r)$ = T , i.e. § holds.
  - Since § holds whenever * holds, we have entailment.

For every valuation ν such that

ν satisfies each of $\varphi_1, \ldots, \varphi_n$, (*)

ν also satisfies ψ. (§)

Validity and Tautology

- $\models \psi$ is the special case for n = 0, and we say ψ is **valid.**

  Every valuation must induce T for ψ, because every valuation vacuously induces T for every formula on the LHS.

  (For the propositional case, we can also say ψ is a **tautology**. For the predicate logic case, not every valid formula is a tautology, although some are.)

- $\vdash \psi$ is the special case for n = 0, meaning that ψ is **provable** from the **empty set** of premises.

Satisfying a *Set* of Formulas

- Generally, Γ is a (possibly-infinite) **set** of formulas
- A valuation ν **satisfies** Γ

  iff ν satisfies **each** formula in Γ.

Validity vs. Provability

- Generally, Γ is a (possibly-infinite) **set** of formulas
- The symbols $\vdash$ and $\models$ are part of the meta-language.
- $\Gamma \vdash \psi$ means ψ is **provable** from formulas Γ
- $\Gamma \models \psi$ means: Every valuation that satisfies Γ also satisfies ψ.

Satisfiability of a *Set* of Formulas

- Set Γ is **satisfiable** if there is a valuation that satisfies it.
- **Lemma S**: Γ is **satisfiable** iff ***not*** ($\Gamma \models \perp$).
- **Proof** follows on next slide.
- **Corollary**: Γ is ***un*satisfiable** iff $\Gamma \models \perp$.

Satisfiability of a *Set* of Formulas

- **Proof**: The following statements are equivalent:
- Γ is satisfiable.
- Γ is **satisfied** by some ν.
- Γ is **satisfied** by some ν that does *not* satisfy $\perp$ (because no valuation satisifies $\perp$).
- not ($\Gamma \models \perp$).

Soundness vs. Completeness of a Logical System (such as ND)

- **Soundness**: Every provable sequent is an entailment:

  (for every set Γ and formula ψ):

  $\Gamma \vdash \psi$ implies $\Gamma \models \psi$

- **Completeness**: Every entailment is provable:

  (for every set Γ and formula ψ):

  $\Gamma \models \psi$ implies $\Gamma \vdash \psi$

Proof of Soundness

- **Soundness**: Every sequent of Natural Deduction is an entailment:

  for every Γ, ψ:

  $$\Gamma \vdash \psi \text{ implies } \Gamma \models \psi$$

- Assume that $\Gamma \vdash \psi$, to show $\Gamma \models \psi$.
- This will be by **structural induction** on the **proof tree** of ψ from formulas in Γ.
- In order to make this as clean as possible, we develop another way of representing proof trees, called **contextual representation**.

Contextual Representation of Natural Deduction Rules

- In the previous representation of natural deduction rules, the **full context** of antecedents is **implicit**.

- For example, with $\wedge$ Introduction, formulas that lead to φ and ψ in the proof are not shown explicitly.

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \qquad \wedge I$$

- For the soundness proof, however, it will be helpful to show the context of each formula explicitly.

- So we **restate** this rule **with context** (**sets of formulas** Γ, Δ) as follows:

$$\frac{\Gamma \vdash \varphi \qquad \Delta \vdash \psi}{\Gamma \cup \Delta \vdash \varphi \wedge \psi} \qquad \wedge I$$

Advantage of Contextual Representation of Natural Deduction Rules

The contextual form will have advantages when **temporary assumptions** are involved, such as in the $\rightarrow$I rule:

$$\frac{\Gamma \cup \{\varphi\} \vdash \psi}{\Gamma \vdash \varphi \rightarrow \psi} \quad \rightarrow\mathrm{I}$$

This expresses the fact that, in implies-introduction proving $\varphi \rightarrow \psi$, **the context for the proof of ψ inside the box consists of the context outside the box**, augmented with the assumption φ.

Natural Deduction Rules in Contextual Form		
	Introduction	**Elimination**
$\wedge$	$\dfrac{\Gamma \vdash \varphi \qquad \Delta \vdash \psi}{\Gamma \cup \Delta \vdash \varphi \wedge \psi}$	$\dfrac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi} \qquad \dfrac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi}$
$\vee$	$\dfrac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi} \qquad \dfrac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}$	$\dfrac{\Gamma \vdash \varphi \vee \psi \quad \Delta \cup \{\varphi\} \vdash \xi \quad \Omega \cup \{\psi\} \vdash \xi}{\Gamma \cup \Delta \cup \Omega \vdash \xi}$
$\rightarrow$	$\dfrac{\Gamma \cup \{\varphi\} \vdash \psi}{\Gamma \vdash \varphi \rightarrow \psi}$	$\dfrac{\Gamma \vdash \varphi \qquad \Delta \vdash \varphi \rightarrow \psi}{\Gamma \cup \Delta \vdash \psi}$
$\neg$	$\dfrac{\Gamma \cup \{\varphi\} \vdash \perp}{\Gamma \vdash \neg \varphi}$	$\dfrac{\Gamma \vdash \varphi \qquad \Delta \vdash \neg \varphi}{\Gamma \cup \Delta \vdash \perp}$
RAA $\perp$E	$\dfrac{\Gamma \cup \{\neg \varphi\} \vdash \perp}{\Gamma \vdash \varphi}$	$\dfrac{}{\Gamma \cup \{\perp\} \vdash \varphi}$
Ax	$\dfrac{}{\Gamma \vdash \top} \qquad \dfrac{}{\Gamma \cup \{\varphi\} \vdash \varphi}$	

Note: This form is related to Gentzen's "Sequent Calculus".

Example of a Proof with Contexts

Build tree from the bottom up

Lines with nothing above are Ax

Rules

$$\dfrac{\dfrac{}{\{E\vee F\} \vdash E\vee F} \qquad \dfrac{}{\{E\} \vdash E} \qquad \dfrac{\dfrac{\dfrac{}{\{\neg F\} \vdash \neg F} \qquad \dfrac{}{\{F\} \vdash F}}{\{\neg F, F\} \vdash \bot}\ \neg E}{\{\neg F, F\} \vdash E}\ \bot E}{\{E\vee F, \neg F\} \vdash E}\ \vee E$$

1: $E\vee F$, $\neg F$ premises

2: $\boxed{E}$ assumption

3: F assumption

4: $\bot$ $\neg$ elim 3,1.2

5: E contra (constructive) 4

6: E $\vee$ elim 1.1,2-2,3-5

Example of Box, Tree, and Contextual Forms for the same proof

Box

1: $(E \land F) \to G$ premise

2: E assumption

3: F assumption

4: $E \land F$ $\land$ intro 2,3

5: G $\to$ elim 1,4

6: $F \to G$ $\to$ intro 3-5

7: $E \to F \to G$ $\to$ intro 2-6

Tree

$$\dfrac{\dfrac{\dfrac{\dfrac{[E]_1 \qquad [F]_2}{E \land F}\,\land I \qquad E \land F \to G}{G}\,\to E}{F \to G}\,\to I_2}{E \to (F \to G)}\,\to I_1$$

Contextual

(Note that leaves can be premises or assumptions. Discharge is implicit.)

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\{E\} \vdash E} \qquad \overline{\{F\} \vdash F}}{\{E, F\} \vdash E \land F}\,\land I \qquad \overline{\{E \land F \to G\} \vdash E \land F \to G}}{\{E \land F \to G, E, F\} \vdash G}\,\to E}{\{E \land F \to G, E\} \vdash F \to G}\,\to I}{\{E \land F \to G\} \vdash E \to (F \to G))}\,\to I$$

$\{E \land F \to G, E, F\} \vdash G$ $\to E$ (note LHS union)

$\{E \land F \to G, E\} \vdash F \to G$ $\to I$ (note reduction in context)

$\{E \land F \to G\} \vdash E \to (F \to G)$ $\to I$ (note reduction in context)

Contextual Rule Applications (justification is *below* the lines)

Step by Step

Box

Tree

1: $(E \wedge F) \rightarrow G$ premise

...

2: $E \rightarrow F \rightarrow G$

$E \wedge F \rightarrow G$

...

$E \rightarrow (F \rightarrow G)$

Contextual

$\{E \wedge F \rightarrow G\} \vdash E \rightarrow (F \rightarrow G)$

Step by Step

Box

1: $(E \wedge F) \to G$ premise

2: E assumption

. . .

3: $F \to G$

4: $E \to F \to G$ $\to$ intro 2-3

Tree

$$\dfrac{\begin{matrix}[E]_1 & E \wedge F \to G \\ & \vdots \\ F \to G & \end{matrix}}{E \to (F \to G)} \;\to I_1$$

Contextual

$$\dfrac{\{E \wedge F \to G,\ E\} \vdash F \to G}{\{E \wedge F \to G\} \vdash E \to (F \to G)} \;\to I$$

Step by Step

Box

1: $(E \wedge F) \to G$ premise

2: E assumption

3: F assumption

. . .

4: G

5: $F \to G$ $\to$ intro 3–4

6: $E \to F \to G$ $\to$ intro 2–5

Tree

$$\dfrac{\dfrac{\dfrac{[E]_1 \quad [F]_2 \quad \dots \quad E \wedge F \to G}{G}\ \to I_2}{F \to G}\ \to I_1}{E \to (F \to G)}$$

Contextual

$$\dfrac{\dfrac{\{E \wedge F \to G,\ E,\ F\} \vdash G}{\{E \wedge F \to G,\ E\} \vdash F \to G}\ \to I}{\{E \wedge F \to G\} \vdash E \to (F \to G)}\ \to I$$

Step by Step

Box

1:	$(E\wedge F)\to G$	premise
2:	E	assumption
3:	F	assumption
	$\ldots$	
4:	$E\wedge F$	
5:	G	$\to$ elim 1,4
6:	$F\to G$	$\to$ intro 3–5
7:	$E\to F\to G$	$\to$ intro 2–6

Tree

$$\dfrac{\dfrac{\dfrac{\begin{matrix}[E]_1 \qquad [F]_2 \\ E\wedge F\end{matrix} \qquad E\wedge F\to G}{G}\;\to E}{F\to G}\;\to I_2}{E\to(F\to G)}\;\to I_1$$

Contextual

$$\dfrac{\dfrac{\dfrac{\{E\wedge F\to G,\ E,\ F\}\ \vdash E\wedge F \qquad \{E\wedge F\to G\}\ \vdash E\wedge F\to G}{\{E\wedge F\to G,\ E,\ F\}\ \vdash G}\;\to E}{\{E\wedge F\to G,\ E\}\ \vdash F\to G}\;\to I}{\{E\wedge F\to G\}\ \vdash E\to(F\to G)}\;\to I$$

Example of Box, Tree, and Contextual Forms for the same proof

Box

1:	$(E \wedge F) \rightarrow G$	premise
2:	E	assumption
3:	F	assumption
4:	$E \wedge F$	$\wedge$ intro 2,3
5:	G	$\rightarrow$ elim 1,4
6:	$F \rightarrow G$	$\rightarrow$ intro 3-5
7:	$E \rightarrow F \rightarrow G$	$\rightarrow$ intro 2-6

Tree

$$
\cfrac{\cfrac{\cfrac{\cfrac{[E]_1 \qquad [F]_2}{E \wedge F}\;\wedge I \qquad E \wedge F \rightarrow G}{G}\;\rightarrow E}{F \rightarrow G}\;\rightarrow I_2}{E \rightarrow (F \rightarrow G)}\;\rightarrow I_1
$$

Contextual

(Note that leaves can be premises or assumptions. Discharge is implicit.)

$$
\cfrac{\cfrac{\cfrac{\cfrac{}{\{E\} \vdash E} \qquad \cfrac{}{\{F\} \vdash F}}{\{E, F\} \vdash E \wedge F}\;\wedge I \qquad \cfrac{}{\{E \wedge F \rightarrow G\} \vdash E \wedge F \rightarrow G}}{\{E \wedge F \rightarrow G, E, F\} \vdash G}\;\rightarrow E}{\cfrac{\{E \wedge F \rightarrow G, E\} \vdash F \rightarrow G}{\{E \wedge F \rightarrow G\} \vdash E \rightarrow (F \rightarrow G)}\;\rightarrow I}\;\rightarrow I
$$

$\rightarrow E$ (note LHS union)

$\rightarrow I$ (note reduction in context)

$\rightarrow I$ (note reduction in context)

Contextual Rule Applications (justification is *below* the lines)

Proof of Soundness

- We are proving: $\Gamma \vdash \psi$ implies $\Gamma \models \psi$.
- We will show by **structural induction** that in the nodes of **any** contextual tree formed by following the rules of inference, we can **replace $\vdash$ with $\models$**.
- For example:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\{E\} \vdash E} \quad \dfrac{}{\{F\} \vdash F}}{\{E, F\} \vdash E \wedge F}\,\wedge I \qquad \dfrac{}{\{E \wedge F \to G\} \vdash E \wedge F \to G}}{\{E \wedge F \to G, E, F\} \vdash G}\,\to E}{\{E \wedge F \to G, E\} \vdash F \to G}\,\to I}{\{E \wedge F \to G\} \vdash E \to (F \to G)}\,\to I
$$

becomes

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\{E\} \models E} \quad \dfrac{}{\{F\} \models F}}{\{E, F\} \models E \wedge F} \qquad \dfrac{}{\{E \wedge F \to G\} \models E \wedge F \to G}}{\{E \wedge F \to G, E, F\} \models G}}{\{E \wedge F \to G, E\} \models F \to G}}{\{E \wedge F \to G\} \models E \to (F \to G)}
$$

Proof of Soundness

- We are proving: $\Gamma \vdash \psi$ implies $\Gamma \models \psi$, i.e.
  if there is a proof of ψ from Γ,
  then for any valuation ν such that $\nu(\Gamma)$ = T, also $\nu(\psi)$ = T.

- **Structural induction on the tree of the proof in contextual form.**

- The tree is either a single node, or a node combining one or more sub-trees.

- **Basis**: The simplest proof is a tree of **one node**. From the table, it must then be one of these:
  - $\Gamma \vdash \top$
  - $\Gamma \cup \{\varphi\} \vdash \varphi$
  - $\Gamma \cup \{\bot\} \vdash \varphi$
- In the first case, $\nu(\top)$ = T for any ν, thus $\Gamma \models \top$.

- In the second case, if ν satisfies $\Gamma \cup \{\varphi\}$ then $\nu(\varphi)$ = T, thus $\Gamma \models \varphi$.

- In the third case, ν can't satisfy $\Gamma \cup \{\bot\}$, so $\Gamma \cup \{\bot\} \models \varphi$ vacuously.

Proof of Soundness Continued

- **Induction Step: Adding a root combining one or more subtrees.**
- Suppose that all of the antecedents of a rule in contextual form satisfy the property. We need to show that the consequent satisfies the property as well.
- Example: $\wedge$ Introduction rule:

$$\frac{\Gamma \vdash \varphi \qquad \Delta \vdash \psi}{\Gamma \cup \Delta \vdash \varphi \wedge \psi}$$

- **The induction hypothesis** is that $\Gamma \vdash \varphi$ implies $\Gamma \models \varphi$, and $\Delta \vdash \psi$ implies $\Delta \models \psi$.
- We must show that $\Gamma \cup \Delta \vdash \varphi \wedge \psi$ implies $\Gamma \cup \Delta \models \varphi \wedge \psi$.
- Assume $\Gamma \cup \Delta \vdash \varphi \wedge \psi$, to show $\Gamma \cup \Delta \models \varphi \wedge \psi$.
- Suppose ν satisfies $\Gamma \cup \Delta$, to show $\nu(\varphi \wedge \psi) = T$.
  Then ν satisfies both Γ and Δ.
  From the induction hypotheses, $\nu(\varphi) = \nu(\psi) = T$.
  Thus from the truth table for $h_\wedge$, $\nu(\varphi \wedge \psi) = T$.

Proof of Soundness Continued

- **Induction Step, Continued:**

- The steps for $\wedge$E, $\vee$I, $\rightarrow$E, $\neg$E, $\perp$E (rules that don't introduce assumptions) are analogous to that for $\wedge$I, and are left to the reader.

Proof of Soundness Continued

- **Induction Step, Continued:**
- Example: $\to$ Introduction rule

  $$\frac{\Gamma \cup \{\varphi\} \vdash \psi}{\Gamma \vdash \varphi \to \psi}$$

- (Here φ is the assumption used in natural deduction, which is discharged at the end of the sub-proof.)
- **The induction hypothesis** is: if $\Gamma \cup \{\varphi\} \vdash \psi$ then $\Gamma \cup \{\varphi\} \models \psi$.
- We must show the conclusion: if $\Gamma \vdash \varphi \to \psi$ then $\Gamma \models \varphi \to \psi$.
- Suppose $\Gamma \vdash \varphi \to \psi$, to show $\Gamma \models \varphi \to \psi$.
- From $\to$I, we know $\Gamma \cup \{\varphi\} \vdash \psi$. So by the induction hypothesis, $\Gamma \cup \{\varphi\} \models \psi$.
- Now suppose that v satisfies Γ. There are two cases:
  - **If $v(\varphi)$ = T**, then v satisfies $\Gamma \cup \{\varphi\}$, and from the induction hypothesis, $v(\psi)$ = T, so $v(\varphi \to \psi)$ = T, from the truth table for $h_{\to}$.
  - **If $v(\varphi)$ = F**, then $v(\varphi \to \psi)$ = T, also from the truth table for $h_{\to}$.
- The step for $\vee$E (which also introduces assumptions) is analogous to the above.

Proof of Soundness Continued

- **Induction Step:**
- RAA

$$\frac{\Gamma \cup \{\neg\varphi\} \vdash \bot}{\Gamma \vdash \varphi}$$

- **The induction hypothesis** is that if $\Gamma \cup \{\neg\varphi\} \vdash \bot$ then $\Gamma \cup \{\neg\varphi\} \models \bot$.
- To be shown is $\Gamma \vdash \varphi$ implies $\Gamma \models \varphi$.
- Suppose $\Gamma \vdash \varphi$. Then by the $\neg$E rule, $\Gamma \cup \{\neg\varphi\} \vdash \bot$, and by the induction hypotheis $\Gamma \cup \{\neg\varphi\} \models \bot$.
- We must show that if ν satisfies Γ then $\nu(\varphi)$ = T.
  - Suppose ν satisfies Γ. Then by lemma S, ν **cannot** satisfy $\Gamma \cup \{\neg\varphi\}$.
  - So we must have $\nu(\neg\varphi)$ = F.
  - But then $\nu(\varphi)$ = T, from the truth table for $h_{\neg}$.
- The proof step for $\neg$I is analogous to the above.
- This concludes the proof of the induction step, and **thus ND is sound**.

Uses of Soundness

- There are **algorithms** for determining whether or not

$$\varphi_1, \ldots, \varphi_n \models \psi$$

- Thus, one can compute a **necessary** condition of whether there is a proof of

$$\varphi_1, \ldots, \varphi_n \vdash \psi$$

- In other words, before embarking on trying to find a proof of a formula, we could check whether the formula follows on semantic grounds first.

Completeness

- Completeness says

  (for all Γ, ψ)

  $$\Gamma \models \psi \text{ implies } \Gamma \vdash \psi$$

- The general case (where Γ could be infinite) will require a "non-constructive" proof.
- The case of Γ **finite** is special, and admits a constructive, even algorithmic, proof.

Finite Completeness

- Finite completeness says (for all $\varphi_1, \ldots, \varphi_n, \psi$)

  $$\varphi_1, \ldots, \varphi_n \models \psi$$

  implies

  $$\varphi_1, \ldots, \varphi_n \vdash \psi$$

- **If** this could be established, then the algorithm mentioned for soundness would be a necessary and **sufficient** condition for the existence of a proof. Thus provability could be testable algorithmically.

- Our proof will use LEM, i.e. it applies to a classical rather than an intuitionistic system.

Proof of Finite Completeness
(following Huth and Ryan)

Three steps are used to show

$\varphi_1, \ldots, \varphi_n \models \psi$ implies $\varphi_1, \ldots, \varphi_n \vdash \psi$:

1. $\varphi_1, \ldots, \varphi_n \models \psi$ implies $\models (\varphi_1 \rightarrow (\varphi_2 \rightarrow \ldots (\varphi_n \rightarrow \psi)) \ldots)$

2. For any formula η, $\models \eta$ implies $\vdash \eta$.
   [η could be $(\varphi_1 \rightarrow (\varphi_2 \rightarrow \ldots (\varphi_n \rightarrow \psi)) \ldots)$, for example.]

3. $\vdash (\varphi_1 \rightarrow (\varphi_2 \rightarrow \ldots (\varphi_n \rightarrow \psi)) \ldots)$ implies $\varphi_1, \ldots, \varphi_n \vdash \psi$

Step 2 is the key one, as only it bridges the gap between $\models$ and $\vdash$. The other two are simplifying steps, showing that we don't need to worry about the LHS of the turnstiles.

Steps 1 and 3 can be proved by induction on n. I leave them to you.

Proof that for all η

$$\models \eta \text{ implies } \vdash \eta$$

- Assume $\models \eta$. Let $p_1, p_2, \ldots, p_k$ be the set of all proposition symbols that occur in η.

- For each combination of proposition symbols with and without negation, we show that there is a sequent with that combination on the left and the formula of interest on the right:
  - $p_1, \quad p_2, \ldots, p_k \vdash \eta$
  - $\neg p_1, \quad p_2, \ldots, p_k \vdash \eta$
  - $p_1, \neg p_2, \ldots, p_k \vdash \eta$
  - $\neg p_1, \neg p_2, \ldots, p_k \vdash \eta$ etc.

- Then those sequents will be combined into a single sequent of the required form using LEM and $\vee$E.

The Combination Process

- Because this constructs a derivation that is of length exponential in k, we will show it by example, for k = 2.
- Given that we have:
  - $p_1, \; p_2 \vdash \eta$
  - $\neg p_1, \; p_2 \vdash \eta$
  - $p_1, \neg p_2 \vdash \eta$
  - $\neg p_1, \neg p_2 \vdash \eta$
- The proof constructed for the single sequent is shown on the next page.

Proof Constructed for the Single Sequent

1.	$p_1 \vee \neg p_1$	LEM
2.	p_1	Assumption
3.	$p_2 \vee \neg p_2$	LEM
4.	p_2	Assumption
	. . . steps in the proof of $\mathbf{p_1, p_2} \vdash \eta$	
5.	η	
6.	$\neg p_2$	Assumption
	. . . steps in the proof of $\mathbf{p_1, \neg p_2} \vdash \eta$	
7.	η	
8.	η	$\vee$E 3, 4-5, 6-7
9.	$\neg p_1$	Assumption
10.	$p_2 \vee \neg p_2$	LEM
11.	p_2	Assumption
	. . . steps in the proof of $\mathbf{\neg p_1, p_2} \vdash \eta$	
12.	η	
13.	$\neg p_2$	Assumption
	. . . steps in the proof of $\mathbf{\neg p_1, \neg p_2} \vdash \eta$	
14.	η	
15.	η	$\vee$E 10, 11-12, 13-14
16.	η	$\vee$E 1, 2-8, 9-15

Proofs for the Individual Sequents

We want to show that **for any formula** η,

if $\models \eta$

then each of the individual sequents below has a proof

- $p_1, \; p_2, \ldots, p_k \vdash \eta$
- $\neg p_1, \; p_2, \ldots, p_k \vdash \eta$
- $p_1, \neg p_2, \ldots, p_k \vdash \eta$
- $\neg p_1, \neg p_2, \ldots, p_k \vdash \eta$ etc. (for every combination of symbols and their negations)

where $p_1, p_2, \ldots, p_k$ are the proposition symbols in η.

Approach: Use **structural induction** on the **structure of the formula** η (rather than on the proof tree as before).

Structural Induction on Formulas

- We'd like to show ***something like***:

  if η is constructed of sub-formulas, say $\varphi \circ \psi$ where $\circ$ is some connective,

  then for any combination $p^*_1, p^*_2, \ldots, p^*_k$ of negated and un-negated proposition symbols: if

  $p^*_1, p^*_2, \ldots, p^*_k \vdash \varphi$ and

  $p^*_1, p^*_2, \ldots, p^*_k \vdash \psi$ then

  $p^*_1, p^*_2, \ldots, p^*_k \vdash \varphi \circ \psi$.

- However this is not always true
  (for example in the case of $\circ$ being $\neg$).

Note on Inductive Proofs in General

- In many cases in CS and Math, when faced with an inductive proof, we have to prove a **stronger** statement than the one we wish to "take away".

- This is because the induction hypothesis supplied by the take-away statement itself is **too weak** to enable the inductive conclusion to be drawn. We need a stronger inductive hypothesis, and it has to be able to deduce a stronger conclusion at the same time.

- The present situation is an example. Be on the lookout for others during your career.

Proofs for the Individual Sequents

- A **revised statement** that will enable structural induction is:

- **For any valuation** ν, let $p^*_1, p^*_2, \ldots, p^*_k$ be the proposition symbols or their negations (depending on ν, e.g. $\neg p_1, p_2, \ldots, \neg p_k$) such that $\{p^*_1, p^*_2, \ldots, p^*_k\}$ is satisfied by ν (e.g. $\nu(p_1) = F$, $\nu(p_2) = T$, etc.)

- **Lemma:** For any formula η and valuation ν:

A(η): **If** $\nu(\eta)$ = **T**, then $p^*_1, p^*_2, \ldots, p^*_k \vdash \eta$.

take-away

B(η): **If** $\nu(\eta)$ = **F**, then $p^*_1, p^*_2, \ldots, p^*_k \vdash (\neg\eta)$.

strengthener

note ¬

Proving $A(\eta)$: If $\nu(\eta) = \mathbf{T}$ then $p^*_1, p^*_2, \ldots, p^*_k \vdash \eta$.

$B(\eta)$: If $\nu(\eta) = \mathbf{F}$ then $p^*_1, p^*_2, \ldots, p^*_k \vdash (\neg\eta)$.

- This is done by **structural induction** on the **structure** of the **formula** η.
- **Basis**: If η is a **single proposition symbol** p, then:
  - If $\nu(p) = \mathbf{T}$, then p* must be **p**, and we certainly have $p \vdash p$ (case A).
  - If $\nu(p) = \mathbf{F}$, then p* must be **¬p**, and we have $\neg p \vdash (\neg p)$ (case B).
- If η is $\top$, then $\nu(\top) = \mathbf{T}$ always, but also $\vdash \top$ (by TI) (case A).
- If η is $\bot$, then $\nu(\bot) = \mathbf{F}$ always, but also $\vdash \neg\bot$ (by ¬I) (case B).

Proving A(η): If $\nu(\eta)$ = **T** then $p^*_1, p^*_2, \ldots, p^*_k \vdash \eta$.

B(η): If $\nu(\eta)$ = **F** then $p^*_1, p^*_2, \ldots, p^*_k \vdash (\neg\eta)$.

- **Induction Step**: We have to show that the inductive hypothesis A(η)and B(η) implies the conclusion for each possible operator: $\neg \wedge \vee \rightarrow$ forming η at the top level.

- For example, if η is $\varphi\vee\psi$, then we show:

if A(φ) and B(φ), and A(ψ) and B(ψ),

then also A($\varphi\vee\psi$) and B($\varphi\vee\psi$).

Fortunately one of A or B is vacuously true for any conclusion η.

Case where η is of form $\neg\rho$ for some ρ :

- Case 1: $\nu(\eta) = \mathbf{T}$
- Then $\nu(\rho) = \mathbf{F}$.
- By the induction hypothesis, $B(\rho)$:

$$p^*_1, p^*_2, \ldots, p^*_k \vdash (\neg\rho),$$

but that is the same as $A(\eta)$:

$$p^*_1, p^*_2, \ldots, p^*_k \vdash \eta$$

Case where η is of form $\neg\rho$ for some ρ :

- Case 2: $\nu(\eta)$ = **F**
- Then $\nu(\rho)$ = **T**.
- By the induction hypothesis, $A(\rho)$:

$$p^*_1, p^*_2, \ldots, p^*_k \vdash \rho.$$

Using derived rule **¬¬I** to extend the proof, we have

$$p^*_1, p^*_2, \ldots, p^*_k \vdash \neg(\neg\rho)$$

Therefore $B(\eta)$:

$$p^*_1, p^*_2, \ldots, p^*_k \vdash \neg\eta$$

Case where η is of form $\rho_1 \wedge \rho_2$:

- We need to consider 4 cases:
  $\nu(\rho_1, \rho_2)$ = **FF**, **FT**, **TF**, and **TT**.

- If $\nu(\rho_1)$ = **F**: in which case $\nu(\rho_1 \wedge \rho_2)$ = **F:**
  By the induction hypothesis
  $p^*_1, p^*_2, \ldots, p^*_k \vdash (\neg\rho_1)$

  Using ND rules, we derive with a few more steps, a proof of
  $p^*_1, p^*_2, \ldots, p^*_k \vdash \neg(\rho_1 \wedge \rho_2)$

  This conforms to case B.

- A similar argument applies if $\nu(\rho_2)$ = **F**.
  So three of four cases have now been covered.

Case where η is of form $\rho_1 \wedge \rho_2$:

- Now address the remaining case.

- If $\nu(\rho_1) = \nu(\rho_2) = \mathbf{T}$**,** we have by the induction hypothesis
  $p^*_1, p^*_2, \ldots, p^*_k \vdash \rho_1$
  $p^*_1, p^*_2, \ldots, p^*_k \vdash \rho_2$

  These proofs can be combined using $\wedge$**I** to get a proof of $\rho_1 \wedge \rho_2$ (case A).

- **The steps for the other operators ($\vee$, $\rightarrow$) are similar.**

Algorithm-Based Proof

- The proof just outlined is sufficiently constructive that we can create an **algorithm** from it:

- Given a valid formula η, **generate** a natural deduction proof of η.

- In some sense, such an algorithmic proof is useful, in that it can be live-tested by computer for various examples, **unlike an ordinary proof**.

Example of Algorithmically-Generated Proof by my prover.pro

DNE: ¬¬p→q

```
?- testTautology(implies(not(not(p)), p)).

Proof for tautology: implies(not(not(p)), p):

| 1: or(p, not(p)) [lem]
| --------------------
| | 2: p [assumption(or-elim)]
| | 3: implies(not(not(p)), p) [implies-intro(2)]
| --------------------
| --------------------
| | 4: not(p) [assumption(or-elim)]
| | 5: not(not(not(p))) [not-not-intro(4)]
| | --------------------
| | | 6: not(not(p)) [assumption(implies-intro)]
| | | 7: bottom [not-elim(5, 6)]
| | | 8: p [bottom(7)]
| | --------------------
| | 9: implies(not(not(p)), p) [implies-intro(6-8)]
| --------------------
| 10: implies(not(not(p)), p) [or-elim(1, 2-3, 4-9)]
```

Example of Algorithmically-Generated Proof by my prover.pro

Peirce's law: $((p \rightarrow q) \rightarrow p) \rightarrow p$

(can be proved by a human using RAA rather than LEM in 12 steps)

```
Proof for tautology: implies(implies(implies(p, q), p), p):

| 1: or(p, not(p)) [lem]
| --------------------
| | 2: p [assumption(or-elim)]
| | 3: or(q, not(q)) [lem]
| | --------------------
| | | 4: q [assumption(or-elim)]
| | | 5: implies(implies(implies(p, q), p), p) [implies-intro(2)]
| | --------------------
| | --------------------
| | | 6: not(q) [assumption(or-elim)]
| | | 7: implies(implies(implies(p, q), p), p) [implies-intro(2)]
| | --------------------
| | 8: implies(implies(implies(p, q), p), p) [or-elim(3, 4-5, 6-7)]
| --------------------
| --------------------
| | 9: not(p) [assumption(or-elim)]
| | 10: or(q, not(q)) [lem]
| | --------------------
| | | 11: q [assumption(or-elim)]
| | | --------------------
| | | | 12: p [assumption(implies-intro)]
| | | | 13: bottom [not-elim(9, 12)]
| | | | 14: q [bottom(13)]
| | | --------------------
| | | 15: implies(p, q) [implies-intro(12-14)]
| | | --------------------
| | | | 16: implies(implies(p, q), p) [assumption(not-intro)]
| | | | 17: p [implies-elim(15, 16)]
| | | | 18: bottom [not-elim(9, 17)]
| | | --------------------
| | | 19: not(implies(implies(p, q), p)) [not-intro(16-18)]
```

```
| | | --------------------
| | | | 20: implies(implies(p, q), p) [assumption(implies-intro)]
| | | | 21: bottom [not-elim(19, 20)]
| | | | 22: p [bottom(21)]
| | | --------------------
| | | 23: implies(implies(implies(p, q), p), p) [implies-intro(20-22)]
| | --------------------
| | --------------------
| | | 24: not(q) [assumption(or-elim)]
| | | --------------------
| | | | 25: p [assumption(implies-intro)]
| | | | 26: bottom [not-elim(9, 25)]
| | | | 27: q [bottom(26)]
| | | --------------------
| | | 28: implies(p, q) [implies-intro(25-27)]
| | | --------------------
| | | | 29: implies(implies(p, q), p) [assumption(not-intro)]
| | | | 30: p [implies-elim(28, 29)]
| | | | 31: bottom [not-elim(9, 30)]
| | | --------------------
| | | 32: not(implies(implies(p, q), p)) [not-intro(29-31)]
| | | --------------------
| | | | 33: implies(implies(p, q), p) [assumption(implies-intro)]
| | | | 34: bottom [not-elim(32, 33)]
| | | | 35: p [bottom(34)]
| | | --------------------
| | | 36: implies(implies(implies(p, q), p), p) [implies-intro(33-35)]
| | --------------------
| | 37: implies(implies(implies(p, q), p), p) [or-elim(10, 11-23, 24-36)]
| --------------------
| 38: implies(implies(implies(p, q), p), p) [or-elim(1, 2-8, 9-37)]
```

Sketch of Completeness for the *General* (not-necessarily finite) Propositional Case

This section is advanced and may be skipped.

- This sketch follows van Dalen, *Logic and Structure*.
- **Definition**: A set of formulas Γ is **consistent** provided

$$\text{not} \quad \Gamma \vdash \bot.$$

- Note the parallel:
  - **Consistency** of Γ: Not $\Gamma \vdash \bot$.
  - **Satisfiability** of Γ: Not $\Gamma \models \bot$.

Lemma A

- For any Γ, φ

$$\Gamma \mid\!\!- \varphi \qquad \text{iff} \qquad \Gamma \cup \{\neg\varphi\} \mid\!\!- \bot$$

- Proof:
  - Suppose $\Gamma \mid\!\!- \varphi$. Then also $\Gamma \cup \{\neg\varphi\} \mid\!\!- \varphi$.
    Trivially $\Gamma \cup \{\neg\varphi\} \mid\!\!- \neg\varphi$.
    So $\Gamma \cup \{\neg\varphi\} \mid\!\!- \bot$ by $\neg$E.

  - Suppose $\Gamma \cup \{\neg\varphi\} \mid\!\!- \bot$. Then by RAA, $\Gamma \mid\!\!- \varphi$.

Lemma B

- For any Γ, φ

  $\Gamma \models \varphi$ iff $\Gamma \cup \{\neg\varphi\} \models \bot$.

- Proof: The following statements are equivalent:
  - $\Gamma \models \varphi$.
  - If ν is a valuation satisfying Γ then satisfies φ.
  - If ν is a valuation satisfying Γ then ν doesn't satisfy $\neg\varphi$.
  - There is no valuation satisfying $\Gamma \cup \{\neg\varphi\}$.
  - Every valuation satisfying $\Gamma \cup \{\neg\varphi\}$ satisfies $\bot$ (vacuously).
  - $\Gamma \cup \{\neg\varphi\} \models \bot$.

Lemma C

- The following are equivalent:
  a) Completeness.
  b) For all Γ, φ, $\Gamma \models \varphi$ implies $\Gamma \vdash \varphi$.
  c) For all Γ, $\Gamma \models \bot$ implies $\Gamma \vdash \bot$.
  d) For all Γ, not $\Gamma \vdash \bot$ implies not $\Gamma \models \bot$.
  e) For all Γ, Γ is consistent implies Γ is satisfiable by some valuation ("Γ has a model").

- Proof:
  - (b) is a restatement of (a).
  - (c) iff (b) is by Lemmas A and B.
  - (d) is the contrapositive of (c).
  - (e) is a restatement of (d).

General Completeness Theorem

- We have shown that completeness is equivalent to:

- (For all Γ)
  Γ consistent implies Γ satisfiable.

- "Every consistent set of formulas has a model."

- Sketch of the proof of the above statement:

  We start with a Γ_0 that is consistent, to eventually show there exists a valuation satisfying Γ_0, based on ND rules.

Sketch, continued

- First extend Γ_0 to a **maximally consistent** set Γ_{max}:
  - **Enumerate** every possible propositional formula $\varphi_0, \varphi_1, \varphi_2, \ldots$ defining sets $\Gamma_0, \Gamma_1, \Gamma_2, \ldots$ as follows:
    - If $\Gamma_i \cup \{\varphi_i\}$ is consistent, Γ_{i+1} is defined as $\Gamma_i \cup \{\varphi_i\}$.

      Otherwise Γ_{i+1} is defined as Γ_i.
  - (The Axiom of Choice is being used here.)
  - The **limit** of this process is $\cup\{\Gamma_0, \Gamma_1, \Gamma_2, \ldots\} = \Gamma_{max}$.
- Then show that Γ_{max} is consistent, and in fact, maximally consistent.

Sketch, continued

- $\Gamma_{\mathbf{max}}$ is ***consistent***, because at no step is a formula added that would destroy its consistency.

- It is ***maximally*** consistent because it can be shown to be **closed under derivability**:
  If $\Gamma_{max} \vdash \varphi$, then in fact $\varphi \in \Gamma_{max}$.

- We then show that any maximally consistent set has an valuation satisfying it. **Define such a valuation** ν as follows:

  - For each proposition symbol p, if p $\in \Gamma_{max}$ then ν(p) = T, otherwise ν(p) = F.

- Then argue that ν **satisfies** Γ_{max} using closure under derivability (using a soundness-like argument).

- Finally, ν also satisfies Γ_0, since $\Gamma_0 \subseteq \Gamma_{max}$. So Γ_0 is satisfiable.