



Semantics, Soundness, and Completeness for Propositional Natural Deduction

Robert Keller
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Proof vs. Truth

- So far, we have seen a method (natural deduction) for **proof** of formulas.
- It would be nice if we had an **independent** definition of **truth** of those formulas so that we could ascertain whether
 - Our proofs are proving only *true* statements. (**soundness**)
 - There is nothing lacking in our proof system. (**completeness**)



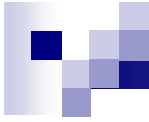
Giving Formulas a Meaning

- An **valuation** is a function v (Greek “nu”) that associates a value in $\{T, F\}$ to every proposition symbol, with the requirement that, for the special symbols \top and \perp (“top” and “bottom”):
 - $v(\top) = T$
 - $v(\perp) = F$
- In the range of v , T is intended to represent “true” and F “false”.
- A valuation is variously called an *assignment*, *interpretation*, or (in CS) *environment* (depending on the author or text).

Induced Values for Formulas

- An **valuation** v **induces** a value $v(\varphi)$ in $\{T, F\}$ in any formula φ , inductively as follows:

Formula φ	Value $v(\varphi)$	
single proposition symbol p	$v(p)$ as given by the valuation v	
$\neg F$	$h_{\neg}(v(F))$	where $h_{\neg}(F) = T$ and $h_{\neg}(T) = F$
$F \wedge G$	$h_{\wedge}(v(F), v(G))$	where $h_{\wedge}(x, y) = T$ iff $x = T$ and $y = T$
$F \vee G$	$h_{\vee}(v(F), v(G))$	where $h_{\vee}(x, y) = T$ iff $x = T$ or $y = T$
$F \rightarrow G$	$h_{\rightarrow}(v(F), v(G))$	where $h_{\rightarrow}(x, y) = T$ iff $x = F$ or $y = T$
$F \leftrightarrow G$	$h_{\leftrightarrow}(v(F), v(G))$	where $h_{\leftrightarrow}(x, y) = T$ iff $x = y$



Truth Function Summary

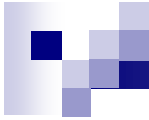
x	y	$h_{\neg}(y)$	$h_{\wedge}(x, y)$	$h_{\vee}(x, y)$	$h_{\rightarrow}(x, y)$	$h_{\leftrightarrow}(x, y)$
F	F	T	F	F	T	T
F	T	F	F	T	T	F
T	F		F	T	F	F
T	T		T	T	T	T

Example of Induced Value

- Formula: $p \vee q \rightarrow \neg p \wedge q$
- Valuation: $v(p) = F, v(q) = T, \dots$
- Induced Value: $v(p \vee q \rightarrow \neg p \wedge q)$
 - $= h_{\rightarrow}(v(p \vee q), v(\neg p \wedge q))$
 - $= h_{\rightarrow}(h_{\vee}(v(p), v(q)), h_{\wedge}(h_{\neg}(v(p)), v(q)))$
 - $= h_{\rightarrow}(h_{\vee}(F, T), h_{\wedge}(h_{\neg}(F), T))$
 - $= h_{\rightarrow}(T, h_{\wedge}(T, T))$
 - $= h_{\rightarrow}(T, T)$
 - $= T$

Another Example of Induced Value

- Formula: $p \vee q \rightarrow \neg p \wedge q$
- Valuation: $v(p) = T, v(q) = F, \dots$
- Induced Value: $v(p \vee q \rightarrow \neg p \wedge q)$
 - $= h_{\rightarrow}(v(p \vee q), v(\neg p \wedge q))$
 - $= h_{\rightarrow}(h_{\vee}(v(p), v(q)), h_{\wedge}(h_{\neg}(v(p)), v(q)))$
 - $= h_{\rightarrow}(h_{\vee}(T, F), h_{\wedge}(h_{\neg}(T), F))$
 - $= h_{\rightarrow}(T, h_{\wedge}(F, F))$
 - $= h_{\rightarrow}(T, F)$
 - $= F$



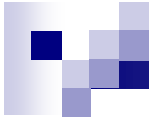
Language Interpreter

- Note that the determination of the induced value of an expression given a valuation is essentially defining an **interpreter** for the language.
- The valuation would typically be called an “environment” in that context.



Satisfaction Definition

- A valuation ν **satisfies** a formula φ iff the induced value $\nu(\varphi) = \text{T}$.
- A formula is **satisfiable** iff there is some valuation that satisfies it. Otherwise it is **unsatisfiable**.
- Examples:
 - $p \rightarrow \neg p$ is satisfiable (for what valuation?)
 - $p \wedge \neg p$ is unsatisfiable



Semantic Entailment: Double Turnstile

- Let $\varphi_1, \dots, \varphi_n, \psi$ be formulas.
- The meaning of

$$\varphi_1, \dots, \varphi_n \models \psi$$

is:

For every valuation v such that

v satisfies each of $\varphi_1, \dots, \varphi_n$, (*)

v also satisfies ψ . (§)

Example of Entailment \models

- Determine whether $p \vee q, \neg q \vee r \models p \vee r$
- We need to look at at most 8 valuations v , one for each possible value of $v(p), v(q), v(r)$.

$v(p)$	$v(q)$	$v(r)$	$v(p \vee q)$	$v(\neg q \vee r)$	* holds (LHS)	§ holds (RHS)	$v(p \vee r)$
F	F	F	F	T			F
F	F	T	F	T		✓	T
F	T	F	T	F			F
F	T	T	T	T	✓	✓	T
T	F	F	T	T	✓	✓	T
T	F	T	T	T	✓	✓	T
T	T	F	T	F		✓	T
T	T	T	T	T	✓	✓	T



Example of \models

- Determine whether or not $p \vee q, \neg q \vee r \models p \vee r$
- **Alternatively**, we could *reason* as follows:
 - $v(q) = F$ or T .
 - If $v(q) = F$, then $*$ holds iff $v(p) = T$, and in that case $v(p \vee r) = T$, i.e. ξ holds.
 - If $v(q) = T$, then $*$ holds iff $v(r) = T$, and in that case $v(p \vee r) = T$, i.e. ξ holds.
 - Since ξ holds whenever $*$ holds, we have entailment.

For every valuation v such that

v satisfies each of $\varphi_1, \dots, \varphi_n$ ($*$)

v also satisfies ψ . (ξ)



Validity and Tautology

- $\models \psi$ is the special case for $n = 0$, and we say ψ is **valid**.

Every valuation must induce T for ψ , because every valuation vacuously induces T for every formula on the LHS.

(For the propositional case, we can also say ψ is a **tautology**. For the predicate logic case, not every valid formula is a tautology, although some are.)

- $\vdash \psi$ is the special case for $n = 0$, meaning that ψ is **provable** from the **empty set** of premises.



Satisfying a *Set* of Formulas

- Generally, Γ is a (possibly-infinite) **set** of formulas
- A valuation ν **satisfies** Γ
iff ν satisfies **each** formula in Γ .



Validity vs. Provability

- Generally, Γ is a (possibly-infinite) **set** of formulas
- The symbols \vdash and \models are part of the meta-language.
- $\Gamma \vdash \psi$ means ψ is **provable** from formulas Γ
- $\Gamma \models \psi$ means: Every valuation that satisfies Γ also satisfies ψ .




Satisfiability of a *Set* of Formulas

- Set Γ is **satisfiable** if there is a valuation that satisfies it.
- **Lemma S:** Γ is **satisfiable** iff **not** $(\Gamma \models \perp)$.
- **Proof** follows on next slide.
- **Corollary:** Γ is **unsatisfiable** iff $\Gamma \models \perp$.



Satisfiability of a *Set* of Formulas

- **Proof:** The following statements are equivalent:
- Γ is satisfiable.
- Γ is **satisfied** by some v .
- Γ is **satisfied** by some v that does *not* satisfy \perp (because no valuation satisfies \perp).
- $\text{not } (\Gamma \models \perp)$.



Soundness vs. Completeness of a Logical System (such as ND)

- **Soundness:** Every provable sequent is an entailment:

(for every set Γ and formula ψ):

$$\Gamma \vdash \psi \text{ implies } \Gamma \models \psi$$

- **Completeness:** Every entailment is provable:

(for every set Γ and formula ψ):

$$\Gamma \models \psi \text{ implies } \Gamma \vdash \psi$$



Proof of Soundness

- **Soundness:** Every sequent of Natural Deduction is an entailment:

for every Γ, ψ :

$$\Gamma \vdash \psi \text{ implies } \Gamma \models \psi$$

- Assume that $\Gamma \vdash \psi$, to show $\Gamma \models \psi$.
- This will be by **structural induction** on the **proof tree** of ψ from formulas in Γ .
- In order to make this as clean as possible, we develop another way of representing proof trees, called **contextual representation**.

Contextual Representation of Natural Deduction Rules

- In the previous representation of natural deduction rules, the **full context** of antecedents is **implicit**.
- For example, with \wedge Introduction, formulas that lead to φ and ψ in the proof are not shown explicitly.

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \quad \wedge I$$

- For the soundness proof, however, it will be helpful to show the context of each formula explicitly.
- So we **restate** this rule **with context** (**sets of formulas** Γ, Δ) as follows:

$$\frac{\Gamma \vdash \varphi \quad \Delta \vdash \psi}{\Gamma \cup \Delta \vdash \varphi \wedge \psi} \quad \wedge I$$

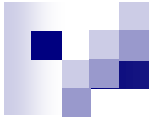


Advantage of Contextual Representation of Natural Deduction Rules

The contextual form will have advantages when **temporary assumptions** are involved, such as in the \rightarrow I rule:

$$\frac{\Gamma \cup \{\varphi\} \vdash \psi}{\Gamma \vdash \varphi \rightarrow \psi} \quad \rightarrow\text{I}$$

This expresses the fact that, in implies-introduction proving $\varphi \rightarrow \psi$, **the context for the proof of ψ inside the box consists of the context outside the box, augmented with the assumption φ .**



Natural Deduction Rules in Contextual Form		
	Introduction	Elimination
\wedge	$\frac{\Gamma \vdash \varphi \quad \Delta \vdash \psi}{\Gamma \cup \Delta \vdash \varphi \wedge \psi}$	$\frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi} \quad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi}$
\vee	$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi} \quad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}$	$\frac{\Gamma \vdash \varphi \vee \psi \quad \Delta \cup \{\varphi\} \vdash \xi \quad \Omega \cup \{\psi\} \vdash \xi}{\Gamma \cup \Delta \cup \Omega \vdash \xi}$
\rightarrow	$\frac{\Gamma \cup \{\varphi\} \vdash \psi}{\Gamma \vdash \varphi \rightarrow \psi}$	$\frac{\Gamma \vdash \varphi \quad \Delta \vdash \varphi \rightarrow \psi}{\Gamma \cup \Delta \vdash \psi}$
\neg	$\frac{\Gamma \cup \{\varphi\} \vdash \perp}{\Gamma \vdash \neg \varphi}$	$\frac{\Gamma \vdash \varphi \quad \Delta \vdash \neg \varphi}{\Gamma \cup \Delta \vdash \perp}$
RAA \perp E	$\frac{\Gamma \cup \{\neg \varphi\} \vdash \perp}{\Gamma \vdash \varphi}$	$\frac{}{\Gamma \cup \{\perp\} \vdash \varphi}$
Ax	$\frac{}{\Gamma \vdash \top} \quad \frac{}{\Gamma \cup \{\varphi\} \vdash \varphi}$	

Note: This form is related to Gentzen's "Sequent Calculus".

Example of Box, Tree, and Contextual Forms for the same proof

Box

1: $(E \wedge F) \rightarrow G$ premise
 2: E assumption
 3: F assumption
 4: $E \wedge F$ \wedge intro 2,3
 5: G \rightarrow elim 1,4
 6: $F \rightarrow G$ \rightarrow intro 3-5
 7: $E \rightarrow F \rightarrow G$ \rightarrow intro 2-6

Tree

$$\frac{\frac{\frac{[E]_1 \quad [F]_2}{E \wedge F} \quad E \wedge F \rightarrow G}{G} \quad \wedge I}{F \rightarrow G} \rightarrow I_2}{E \rightarrow (F \rightarrow G)} \rightarrow I_1$$

Contextual

(Note that leaves can be premises or assumptions. Discharge is implicit.)

$$\frac{\frac{\frac{\frac{\{E\} \vdash E \quad \{F\} \vdash F}{\{E, F\} \vdash E \wedge F} \quad \wedge I}{\{E \wedge F \rightarrow G, E, F\} \vdash G} \rightarrow E}{\{E \wedge F \rightarrow G, E\} \vdash F \rightarrow G} \rightarrow I}{\{E \wedge F \rightarrow G\} \vdash E \rightarrow (F \rightarrow G)} \rightarrow I \quad \text{(note LHS union)}$$

Contextual Rule Applications
 (justification is *below* the lines)

Step by Step

Box

1: $(E \wedge F) \rightarrow G$ premise

...

2: $E \rightarrow F \rightarrow G$

Tree

$E \wedge F \rightarrow G$

...

$E \rightarrow (F \rightarrow G)$

Contextual

$\{E \wedge F \rightarrow G\} \vdash E \rightarrow (F \rightarrow G)$

Step by Step

Box

- 1: $(E \wedge F) \rightarrow G$ premise
- 2: E assumption
- 3: F assumption
- ... (inside box)
- 4: G (inside box)
- 5: $F \rightarrow G$ \rightarrow intro 3-4
- 6: $E \rightarrow F \rightarrow G$ \rightarrow intro 2-5

$$\frac{\{E \wedge F \rightarrow G, E, F\} \vdash G}{\{E \wedge F \rightarrow G, E\} \vdash F \rightarrow G} \rightarrow I$$

$$\{E \wedge F \rightarrow G\} \vdash E \rightarrow (F \rightarrow G) \quad \rightarrow I$$

Tree

$$\frac{\begin{array}{c} [E]_1 \quad \dots \quad [F]_2 \\ \dots \\ G \\ \hline F \rightarrow G \end{array}}{E \rightarrow (F \rightarrow G)} \quad \begin{array}{c} E \wedge F \rightarrow G \\ \rightarrow I_2 \\ \rightarrow I_1 \end{array}$$

Contextual

Step by Step

Box

- 1: $(E \wedge F) \rightarrow G$ premise
- 2: E assumption
- 3: F assumption
- ...
- 4: $E \wedge F$
- 5: G \rightarrow elim 1,4
- 6: $F \rightarrow G$ \rightarrow intro 3-5
- 7: $E \rightarrow F \rightarrow G$ \rightarrow intro 2-6

Tree

$$\begin{array}{r}
 [E]_1 \quad [F]_2 \\
 \frac{\frac{E \wedge F \quad E \wedge F \rightarrow G}{G} \rightarrow E}{F \rightarrow G} \rightarrow I_2 \\
 \frac{F \rightarrow G}{E \rightarrow (F \rightarrow G)} \rightarrow I_1
 \end{array}$$

Contextual

$$\begin{array}{l}
 \frac{\{E \wedge F \rightarrow G, E, F\} \vdash E \wedge F \quad \{E \wedge F \rightarrow G\} \vdash E \wedge F \rightarrow G}{\{E \wedge F \rightarrow G, E, F\} \vdash G} \rightarrow E \\
 \frac{\{E \wedge F \rightarrow G, E, F\} \vdash G}{\{E \wedge F \rightarrow G, E\} \vdash F \rightarrow G} \rightarrow I \\
 \frac{\{E \wedge F \rightarrow G, E\} \vdash F \rightarrow G}{\{E \wedge F \rightarrow G\} \vdash E \rightarrow (F \rightarrow G)} \rightarrow I
 \end{array}$$

Example of Box, Tree, and Contextual Forms for the same proof

Box

- 1: $(E \wedge F) \rightarrow G$ premise
- | | |
|----------------------|-------------------------|
| 2: E | assumption |
| 3: F | assumption |
| 4: $E \wedge F$ | \wedge intro 2,3 |
| 5: G | \rightarrow elim 1,4 |
| 6: $F \rightarrow G$ | \rightarrow intro 3-5 |
- 7: $E \rightarrow F \rightarrow G$ \rightarrow intro 2-6

Tree

$$\begin{array}{c}
 \frac{\frac{[E]_1 \quad [F]_2}{E \wedge F} \quad E \wedge F \rightarrow G}{G} \quad \wedge I}{\frac{F \rightarrow G}{E \rightarrow (F \rightarrow G)}} \rightarrow E \\
 \rightarrow I_2 \\
 \rightarrow I_1
 \end{array}$$

Contextual

(Note that leaves can be premises or assumptions. Discharge is implicit.)

$$\frac{\frac{\frac{\frac{\{E\} \vdash E \quad \{F\} \vdash F}{\{E, F\} \vdash E \wedge F}}{\{E \wedge F \rightarrow G, E, F\} \vdash G} \quad \wedge I}{\{E \wedge F \rightarrow G, E\} \vdash F \rightarrow G} \rightarrow E \quad (note \text{ LHS union})}{\{E \wedge F \rightarrow G\} \vdash E \rightarrow (F \rightarrow G)} \rightarrow I \quad (note \text{ reduction in context})$$

Contextual Rule Applications
(justification is *below* the lines)

Proof of Soundness

- We are proving: $\Gamma \vdash \psi$ implies $\Gamma \models \psi$.
- We will show by **structural induction** that in the nodes of **any** contextual tree formed by following the rules of inference, we can **replace \vdash with \models** .
- For example:

$$\begin{array}{c}
 \frac{\frac{\overline{\{E\} \vdash E} \quad \overline{\{F\} \vdash F}}{\{E, F\} \vdash E \wedge F} \quad \wedge I}{\{E \wedge F \rightarrow G, E, F\} \vdash G} \quad \rightarrow E \\
 \frac{\{E \wedge F \rightarrow G, E\} \vdash F \rightarrow G}{\{E \wedge F \rightarrow G\} \vdash E \rightarrow (F \rightarrow G)} \quad \rightarrow I \\
 \frac{\{E \wedge F \rightarrow G\} \vdash E \rightarrow (F \rightarrow G)}{\{E \wedge F \rightarrow G\} \vdash E \wedge F \rightarrow G}
 \end{array}$$

becomes

$$\begin{array}{c}
 \frac{\frac{\overline{\{E\} \models E} \quad \overline{\{F\} \models F}}{\{E, F\} \models E \wedge F} \quad \wedge I}{\{E \wedge F \rightarrow G, E, F\} \models G} \\
 \frac{\{E \wedge F \rightarrow G, E\} \models F \rightarrow G}{\{E \wedge F \rightarrow G\} \models E \rightarrow (F \rightarrow G)} \\
 \frac{\{E \wedge F \rightarrow G\} \models E \rightarrow (F \rightarrow G)}{\{E \wedge F \rightarrow G\} \models E \wedge F \rightarrow G}
 \end{array}$$



Proof of Soundness

- We are proving: $\Gamma \vdash \psi$ implies $\Gamma \models \psi$, i.e.
if there is a proof of ψ from Γ ,
then for any valuation v such that $v(\Gamma) = T$, also $v(\psi) = T$.
- **Structural induction on the tree of the proof in contextual form.**
- The tree is either a single node, or a node combining one or more sub-trees.
- **Basis:** The simplest proof is a tree of **one node**. From the table, it must then be one of these:
 - $\Gamma \vdash T$
 - $\Gamma \cup \{\varphi\} \vdash \varphi$
 - $\Gamma \cup \{\perp\} \vdash \varphi$
- In the first case, $v(T) = T$ for any v , thus $\Gamma \models T$.
- In the second case, if v satisfies $\Gamma \cup \{\varphi\}$ then $v(\varphi) = T$, thus $\Gamma \models \varphi$.
- In the third case, v can't satisfy $\Gamma \cup \{\perp\}$, so $\Gamma \cup \{\perp\} \models \varphi$ vacuously.



Proof of Soundness Continued

- **Induction Step: Adding a root combining one or more subtrees.**

- Suppose that all of the antecedents of a rule in contextual form satisfy the property. We need to show that the consequent satisfies the property as well.

- Example: \wedge Introduction rule:

$$\frac{\Gamma \vdash \varphi \quad \Delta \vdash \psi}{\Gamma \cup \Delta \vdash \varphi \wedge \psi}$$

- **The induction hypothesis** is that $\Gamma \vdash \varphi$ implies $\Gamma \models \varphi$, and $\Delta \vdash \psi$ implies $\Delta \models \psi$.

- We must show that $\Gamma \cup \Delta \vdash \varphi \wedge \psi$ implies $\Gamma \cup \Delta \models \varphi \wedge \psi$.

- Assume $\Gamma \cup \Delta \vdash \varphi \wedge \psi$, to show $\Gamma \cup \Delta \models \varphi \wedge \psi$.

- Suppose v satisfies $\Gamma \cup \Delta$, to show $v(\varphi \wedge \psi) = \text{T}$.

Then v satisfies both Γ and Δ .

From the induction hypotheses, $v(\varphi) = v(\psi) = \text{T}$.

Thus from the truth table for h_{\wedge} , $v(\varphi \wedge \psi) = \text{T}$.



Proof of Soundness Continued

- **Induction Step, Continued:**
- The steps for $\wedge E$, $\vee I$, $\rightarrow E$, $\neg E$, $\perp E$ (rules that don't introduce assumptions) are analogous to that for $\wedge I$, and are left to the reader.



Proof of Soundness Continued

- **Induction Step, Continued:**

- Example: \rightarrow Introduction rule

$$\frac{\Gamma \cup \{\varphi\} \vdash \psi}{\Gamma \vdash \varphi \rightarrow \psi}$$

- (Here φ is the assumption used in natural deduction, which is discharged at the end of the sub-proof.)
- **The induction hypothesis** is: if $\Gamma \cup \{\varphi\} \vdash \psi$ then $\Gamma \cup \{\varphi\} \models \psi$.
- We must show the conclusion: if $\Gamma \vdash \varphi \rightarrow \psi$ then $\Gamma \models \varphi \rightarrow \psi$.
- Suppose $\Gamma \vdash \varphi \rightarrow \psi$, to show $\Gamma \models \varphi \rightarrow \psi$.
- From $\rightarrow I$, we know $\Gamma \cup \{\varphi\} \vdash \psi$. So by the induction hypothesis, $\Gamma \cup \{\varphi\} \models \psi$.
- Now suppose that v satisfies Γ . There are two cases:
 - **If $v(\varphi) = \mathbf{T}$** , then v satisfies $\Gamma \cup \{\varphi\}$, and from the induction hypothesis, $v(\psi) = \mathbf{T}$, so $v(\varphi \rightarrow \psi) = \mathbf{T}$, from the truth table for h_{\rightarrow} .
 - **If $v(\varphi) = \mathbf{F}$** , then $v(\varphi \rightarrow \psi) = \mathbf{T}$, also from the truth table for h_{\rightarrow} .
- The step for $\forall E$ (which also introduces assumptions) is analogous to the above.

Proof of Soundness Continued

- **Induction Step:**

- RAA

$$\frac{\Gamma \cup \{\neg\varphi\} \vdash \perp}{\Gamma \vdash \varphi}$$

- **The induction hypothesis** is that if $\Gamma \cup \{\neg\varphi\} \vdash \perp$ then $\Gamma \cup \{\neg\varphi\} \models \perp$.
- To be shown is $\Gamma \vdash \varphi$ implies $\Gamma \models \varphi$.
- Suppose $\Gamma \vdash \varphi$. Then by the $\neg E$ rule, $\Gamma \cup \{\neg\varphi\} \vdash \perp$, and by the induction hypothesis $\Gamma \cup \{\neg\varphi\} \models \perp$.
- We must show that if v satisfies Γ then $v(\varphi) = T$.
 - Suppose v satisfies Γ . Then by lemma S, v **cannot** satisfy $\Gamma \cup \{\neg\varphi\}$.
 - So we must have $v(\neg\varphi) = F$.
 - But then $v(\varphi) = T$, from the truth table for \neg .
- The proof step for $\neg I$ is analogous to the above.
- This concludes the proof of the induction step, and **thus ND is sound**.



Uses of Soundness

- There are **algorithms** for determining whether or not

$$\varphi_1, \dots, \varphi_n \models \psi$$

- Thus, one can compute a **necessary** condition of whether there is a proof of

$$\varphi_1, \dots, \varphi_n \vdash \psi$$

- In other words, before embarking on trying to find a proof of a formula, we could check whether the formula follows on semantic grounds first.



Completeness

- Completeness says

(for all Γ, ψ)

$$\Gamma \models \psi \text{ implies } \Gamma \vdash \psi$$

- The general case (where Γ could be infinite) will require a “non-constructive” proof.
- The case of Γ **finite** is special, and admits a constructive, even algorithmic, proof.



Finite Completeness

- Finite completeness says (for all $\varphi_1, \dots, \varphi_n, \psi$)

$$\varphi_1, \dots, \varphi_n \models \psi$$

implies

$$\varphi_1, \dots, \varphi_n \vdash \psi$$

- **If** this could be established, then the algorithm mentioned for soundness would be a necessary and **sufficient** condition for the existence of a proof. Thus provability could be testable algorithmically.
- Our proof will use LEM, i.e. it applies to a classical rather than an intuitionistic system.



Proof of Finite Completeness (following Huth and Ryan)

Three steps are used to show

$$\varphi_1, \dots, \varphi_n \models \psi \text{ implies } \varphi_1, \dots, \varphi_n \vdash \psi :$$

1. $\varphi_1, \dots, \varphi_n \models \psi \text{ implies } \models (\varphi_1 \rightarrow (\varphi_2 \rightarrow \dots (\varphi_n \rightarrow \psi) \dots))$
2. For any formula η , $\models \eta \text{ implies } \vdash \eta$.
[η could be $(\varphi_1 \rightarrow (\varphi_2 \rightarrow \dots (\varphi_n \rightarrow \psi) \dots))$, for example.]
3. $\vdash (\varphi_1 \rightarrow (\varphi_2 \rightarrow \dots (\varphi_n \rightarrow \psi) \dots)) \text{ implies } \varphi_1, \dots, \varphi_n \vdash \psi$

Step 2 is the key one, as only it bridges the gap between \models and \vdash . The other two are simplifying steps, showing that we don't need to worry about the LHS of the turnstiles.

Steps 1 and 3 can be proved by induction on n . I leave them to you.

Proof that for all η

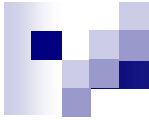
$$|= \eta \text{ implies } \vdash \eta$$

- Assume $|= \eta$. Let p_1, p_2, \dots, p_k be the set of all proposition symbols that occur in η .
- For each combination of proposition symbols with and without negation, we show that there is a sequent with that combination on the left and the formula of interest on the right:
 - $p_1, p_2, \dots, p_k \vdash \eta$
 - $\neg p_1, p_2, \dots, p_k \vdash \eta$
 - $p_1, \neg p_2, \dots, p_k \vdash \eta$
 - $\neg p_1, \neg p_2, \dots, p_k \vdash \eta$ etc.
- Then those sequents will be combined into a single sequent of the required form using LEM and $\vee E$.



The Combination Process

- Because this constructs a derivation that is of length exponential in k , we will show it by example, for $k = 2$.
- Given that we have:
 - $p_1, p_2 \vdash \eta$
 - $\neg p_1, p_2 \vdash \eta$
 - $p_1, \neg p_2 \vdash \eta$
 - $\neg p_1, \neg p_2 \vdash \eta$
- The proof constructed for the single sequent is shown on the next page.



Proof Constructed for the Single Sequent

1.	$p_1 \vee \neg p_1$	LEM
2.	p_1	Assumption
3.	$p_2 \vee \neg p_2$	LEM
4.	p_2	Assumption
	... steps in the proof of $p_1, p_2 \vdash \eta$	
5.	η	
6.	$\neg p_2$	Assumption
	... steps in the proof of $p_1, \neg p_2 \vdash \eta$	
7.	η	
8.	η	$\vee E$ 3, 4-5, 6-7
9.	$\neg p_1$	Assumption
10.	$p_2 \vee \neg p_2$	LEM
11.	p_2	Assumption
	... steps in the proof of $\neg p_1, p_2 \vdash \eta$	
12.	η	
13.	$\neg p_2$	Assumption
	... steps in the proof of $\neg p_1, \neg p_2 \vdash \eta$	
14.	η	
15.	η	$\vee E$ 10, 11-12, 13-14
16.	η	$\vee E$ 1, 2-8, 9-15



Proofs for the Individual Sequents

We want to show that **for any formula** η ,

if $\models \eta$

then each of the individual sequents below has a proof

- $p_1, p_2, \dots, p_k \vdash \eta$
- $\neg p_1, p_2, \dots, p_k \vdash \eta$
- $p_1, \neg p_2, \dots, p_k \vdash \eta$
- $\neg p_1, \neg p_2, \dots, p_k \vdash \eta$ etc. (for every combination of symbols and their negations)

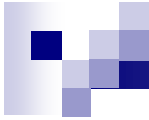
where p_1, p_2, \dots, p_k are the proposition symbols in η .

Approach: Use **structural induction** on the **structure of the formula** η (rather than on the proof tree as before).



Structural Induction on Formulas

- We'd like to show ***something like***:
if η is constructed of sub-formulas, say $\varphi \circ \psi$ where \circ is some connective,
then for any combination $p^*_1, p^*_2, \dots, p^*_k$ of
negated and un-negated proposition symbols:
if
$$p^*_1, p^*_2, \dots, p^*_k \vdash \varphi \text{ and}$$
$$p^*_1, p^*_2, \dots, p^*_k \vdash \psi \text{ then}$$
$$p^*_1, p^*_2, \dots, p^*_k \vdash \varphi \circ \psi.$$
- However this is not always true
(for example in the case of \circ being \neg).



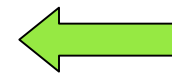
Note on Inductive Proofs in General

- In many cases in CS and Math, when faced with an inductive proof, we have to prove a **stronger** statement than the one we wish to “take away”.
- This is because the induction hypothesis supplied by the take-away statement itself is **too weak** to enable the inductive conclusion to be drawn. We need a stronger inductive hypothesis, and it has to be able to deduce a stronger conclusion at the same time.
- The present situation is an example. Be on the lookout for others during your career.

Proofs for the Individual Sequents

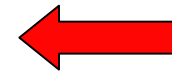
- A **revised statement** that will enable structural induction is:
- **For any valuation** v , let $p^*_1, p^*_2, \dots, p^*_k$ be the proposition symbols or their negations (depending on v , e.g. $\neg p_1, p_2, \dots, \neg p_k$) such that $\{p^*_1, p^*_2, \dots, p^*_k\}$ is satisfied by v (e.g. $v(p_1) = F, v(p_2) = T$, etc.)
- **Lemma:** For any formula η and valuation v :

A(η): **If** $v(\eta) = \mathbf{T}$, then $p^*_1, p^*_2, \dots, p^*_k \vdash \eta$.



take-away


B(η): **If** $v(\eta) = \mathbf{F}$, then $p^*_1, p^*_2, \dots, p^*_k \vdash (\neg\eta)$.



strengthenener




note \neg



Proving $A(\eta)$: If $v(\eta) = \mathbf{T}$ then $p^*_{1}, p^*_{2}, \dots, p^*_{k} \vdash \eta$.

$B(\eta)$: If $v(\eta) = \mathbf{F}$ then $p^*_{1}, p^*_{2}, \dots, p^*_{k} \vdash (\neg\eta)$.

- This is done by **structural induction** on the **structure** of the **formula** η .
- **Basis**: If η is a **single proposition symbol** p , then:
 - If $v(p) = \mathbf{T}$, then p^* must be \mathbf{p} , and we certainly have $p \vdash p$ (case A).
 - If $v(p) = \mathbf{F}$, then p^* must be $\neg\mathbf{p}$, and we have $\neg p \vdash (\neg p)$ (case B).
- If η is \top , then $v(\top) = \mathbf{T}$ always, but also $\vdash \top$ (by TI) (case A).
- If η is \perp , then $v(\perp) = \mathbf{F}$ always, but also $\vdash \neg\perp$ (by \neg I) (case B).



Proving $A(\eta)$: If $v(\eta) = \mathbf{T}$ then $p^*_1, p^*_2, \dots, p^*_k \vdash \eta$.

$B(\eta)$: If $v(\eta) = \mathbf{F}$ then $p^*_1, p^*_2, \dots, p^*_k \vdash (\neg\eta)$.

- **Induction Step:** We have to show that the inductive hypothesis $A(\eta)$ and $B(\eta)$ implies the conclusion for each possible operator: $\neg \wedge \vee \rightarrow$ forming η at the top level.

- For example, if η is $\varphi \vee \psi$, then we show:

if $A(\varphi)$ and $B(\varphi)$, and $A(\psi)$ and $B(\psi)$,

then also $A(\varphi \vee \psi)$ and $B(\varphi \vee \psi)$.

Fortunately one of A or B is vacuously true for any conclusion η .



Case where η is of form $\neg\rho$ for some ρ :

- Case 1: $v(\eta) = \mathbf{T}$
- Then $v(\rho) = \mathbf{F}$.
- By the induction hypothesis, $B(\rho)$:

$$p^*_1, p^*_2, \dots, p^*_k \vdash (\neg\rho),$$

but that is the same as $A(\eta)$:

$$p^*_1, p^*_2, \dots, p^*_k \vdash \eta$$



Case where η is of form $\neg\rho$ for some ρ :

- Case 2: $v(\eta) = \mathbf{F}$
- Then $v(\rho) = \mathbf{T}$.
- By the induction hypothesis, $A(\rho)$:

$$p^*_1, p^*_2, \dots, p^*_k \vdash \rho.$$

Using derived rule $\neg\neg\mathbf{I}$ to extend the proof, we have

$$p^*_1, p^*_2, \dots, p^*_k \vdash \neg(\neg\rho)$$

Therefore $B(\eta)$:

$$p^*_1, p^*_2, \dots, p^*_k \vdash \neg\eta$$



Case where η is of form $\rho_1 \wedge \rho_2$:

- We need to consider 4 cases:
 $v(\rho_1, \rho_2) = \mathbf{FF}, \mathbf{FT}, \mathbf{TF},$ and $\mathbf{TT}.$
- If $v(\rho_1) = \mathbf{F}$: in which case $v(\rho_1 \wedge \rho_2) = \mathbf{F}$:
By the induction hypothesis
 $p^*_1, p^*_2, \dots, p^*_k \vdash (\neg \rho_1)$

Using ND rules, we derive with a few more steps, a proof of
 $p^*_1, p^*_2, \dots, p^*_k \vdash \neg(\rho_1 \wedge \rho_2)$

This conforms to case B.

- A similar argument applies if $v(\rho_2) = \mathbf{F}$.
So three of four cases have now been covered.



Case where η is of form $\rho_1 \wedge \rho_2$:

- Now address the remaining case.
- If $v(\rho_1) = v(\rho_2) = \mathbf{T}$, we have by the induction hypothesis
$$\begin{array}{l} p^*_1, p^*_2, \dots, p^*_k \mid\!-\! \rho_1 \\ p^*_1, p^*_2, \dots, p^*_k \mid\!-\! \rho_2 \end{array}$$

These proofs can be combined using $\wedge\mathbf{I}$ to get a proof of $\rho_1 \wedge \rho_2$ (case A).

- **The steps for the other operators (\vee, \rightarrow) are similar.**



Algorithm-Based Proof

- The proof just outlined is sufficiently constructive that we can create an **algorithm** from it:
- Given a valid formula η , **generate** a natural deduction proof of η .
- In some sense, such an algorithmic proof is useful, in that it can be **live-tested** by computer for various examples, **unlike an ordinary proof.**

Example of Algorithmically-Generated Proof by my prover.pro

DNE: $\neg\neg p \rightarrow q$

```
?- testTautology(implies(not(not(p)), p)).  
Proof for tautology: implies(not(not(p)), p):  
| 1: or(p, not(p)) [lem]  
| -----  
| | 2: p [assumption(or-elim)]  
| | 3: implies(not(not(p)), p) [implies-intro(2)]  
| -----  
| -----  
| | 4: not(p) [assumption(or-elim)]  
| | 5: not(not(not(p))) [not-not-intro(4)]  
| | -----  
| | | 6: not(not(p)) [assumption(implies-intro)]  
| | | 7: bottom [not-elim(5, 6)]  
| | | 8: p [bottom(7)]  
| | -----  
| | 9: implies(not(not(p)), p) [implies-intro(6-8)]  
| -----  
| 10: implies(not(not(p)), p) [or-elim(1, 2-3, 4-9)]
```

Example of Algorithmically-Generated Proof by my prover.pro

Peirce's law: $((p \rightarrow q) \rightarrow p) \rightarrow p$

(can be proved by a human using RAA rather than LEM in 12 steps)

Proof for tautology: `implies(implies(implies(p, q), p), p)`:

```
| 1: or(p, not(p)) [lem]
| -----
| | 2: p [assumption(or-elim)]
| | 3: or(q, not(q)) [lem]
| | -----
| | | 4: q [assumption(or-elim)]
| | | 5: implies(implies(implies(p, q), p), p) [implies-intro(2)]
| | | -----
| | | 6: not(q) [assumption(or-elim)]
| | | 7: implies(implies(implies(p, q), p), p) [implies-intro(2)]
| | | -----
| | | 8: implies(implies(implies(p, q), p), p) [or-elim(3, 4-5, 6-7)]
| | | -----
| | -----
| | 9: not(p) [assumption(or-elim)]
| | 10: or(q, not(q)) [lem]
| | -----
| | | 11: q [assumption(or-elim)]
| | | -----
| | | | 12: p [assumption(implies-intro)]
| | | | 13: bottom [not-elim(9, 12)]
| | | | 14: q [bottom(13)]
| | | | -----
| | | | 15: implies(p, q) [implies-intro(12-14)]
| | | | -----
| | | | 16: implies(implies(p, q), p) [assumption(not-intro)]
| | | | 17: p [implies-elim(15, 16)]
| | | | 18: bottom [not-elim(9, 17)]
| | | | -----
| | | 19: not(implies(implies(p, q), p)) [not-intro(16-18)]
```

```
| | | -----
| | | | 20: implies(implies(p, q), p) [assumption(implies-intro)]
| | | | 21: bottom [not-elim(19, 20)]
| | | | 22: p [bottom(21)]
| | | | -----
| | | | 23: implies(implies(implies(p, q), p), p) [implies-intro(20-22)]
| | | | -----
| | | | 24: not(q) [assumption(or-elim)]
| | | | -----
| | | | | 25: p [assumption(implies-intro)]
| | | | | 26: bottom [not-elim(9, 25)]
| | | | | 27: q [bottom(26)]
| | | | | -----
| | | | 28: implies(p, q) [implies-intro(25-27)]
| | | | -----
| | | | | 29: implies(implies(p, q), p) [assumption(not-intro)]
| | | | | 30: p [implies-elim(28, 29)]
| | | | | 31: bottom [not-elim(9, 30)]
| | | | | -----
| | | | 32: not(implies(implies(p, q), p)) [not-intro(29-31)]
| | | | -----
| | | | | 33: implies(implies(p, q), p) [assumption(implies-intro)]
| | | | | 34: bottom [not-elim(32, 33)]
| | | | | 35: p [bottom(34)]
| | | | | -----
| | | | 36: implies(implies(implies(p, q), p), p) [implies-intro(33-35)]
| | | | -----
| | | 37: implies(implies(implies(p, q), p), p) [or-elim(10, 11-23, 24-36)]
| | | -----
| | 38: implies(implies(implies(p, q), p), p) [or-elim(1, 2-8, 9-37)]
```



Sketch of Completeness for the *General* (not-necessarily finite) Propositional Case **This section is advanced and may be skipped.**

- This sketch follows van Dalen, *Logic and Structure*.
- **Definition:** A set of formulas Γ is **consistent** provided

$$\text{not } \Gamma \vdash \perp.$$

- Note the parallel:
 - **Consistency** of Γ : Not $\Gamma \vdash \perp$.
 - **Satisfiability** of Γ : Not $\Gamma \models \perp$.



Lemma A

- For any Γ, φ

$$\Gamma \vdash \varphi \quad \text{iff} \quad \Gamma \cup \{\neg\varphi\} \vdash \perp$$

- Proof:
 - Suppose $\Gamma \vdash \varphi$. Then also $\Gamma \cup \{\neg\varphi\} \vdash \varphi$.
Trivially $\Gamma \cup \{\neg\varphi\} \vdash \neg\varphi$.
So $\Gamma \cup \{\neg\varphi\} \vdash \perp$ by \neg E.
 - Suppose $\Gamma \cup \{\neg\varphi\} \vdash \perp$. Then by RAA, $\Gamma \vdash \varphi$.



Lemma B

- For any Γ, φ

$$\Gamma \models \varphi \text{ iff } \Gamma \cup \{\neg\varphi\} \models \perp.$$

- Proof: The following statements are equivalent:
 - $\Gamma \models \varphi$.
 - If v is a valuation satisfying Γ then satisfies φ .
 - If v is a valuation satisfying Γ then v doesn't satisfy $\neg\varphi$.
 - There is no valuation satisfying $\Gamma \cup \{\neg\varphi\}$.
 - Every valuation satisfying $\Gamma \cup \{\neg\varphi\}$ satisfies \perp (vacuously).
 - $\Gamma \cup \{\neg\varphi\} \models \perp$.



Lemma C

- The following are equivalent:
 - a) Completeness.
 - b) For all Γ, φ , $\Gamma \models \varphi$ implies $\Gamma \vdash \varphi$.
 - c) For all Γ , $\Gamma \models \perp$ implies $\Gamma \vdash \perp$.
 - d) For all Γ , not $\Gamma \vdash \perp$ implies not $\Gamma \models \perp$.
 - e) For all Γ , Γ is consistent implies Γ is satisfiable by some valuation (“ Γ has a model”).
- Proof:
 - (b) is a restatement of (a).
 - (c) iff (b) is by Lemmas A and B.
 - (d) is the contrapositive of (c).
 - (e) is a restatement of (d).



General Completeness Theorem

- We have shown that completeness is equivalent to:
- (For all Γ)
 Γ consistent implies Γ satisfiable.
- “Every consistent set of formulas has a model.”
- Sketch of the proof of the above statement:

We start with a Γ_0 that is consistent, to eventually show there exists a valuation satisfying Γ_0 , based on ND rules.



Sketch, continued

- First extend Γ_0 to a **maximally consistent** set Γ_{\max} :
 - **Enumerate** every possible propositional formula $\varphi_0, \varphi_1, \varphi_2, \dots$ defining sets $\Gamma_0, \Gamma_1, \Gamma_2, \dots$ as follows:
 - If $\Gamma_i \cup \{\varphi_i\}$ is consistent, Γ_{i+1} is defined as $\Gamma_i \cup \{\varphi_i\}$.
Otherwise Γ_{i+1} is defined as Γ_i .
 - (The Axiom of Choice is being used here.)
 - The **limit** of this process is $\cup\{\Gamma_0, \Gamma_1, \Gamma_2, \dots\} = \Gamma_{\max}$.
- Then show that Γ_{\max} is consistent, and in fact, maximally consistent.



Sketch, continued

- Γ_{\max} is **consistent**, because at no step is a formula added that would destroy its consistency.
- It is **maximally** consistent because it can be shown to be **closed under derivability**:
If $\Gamma_{\max} \vdash \varphi$, then in fact $\varphi \in \Gamma_{\max}$.
- We then show that any maximally consistent set has an valuation satisfying it. **Define such a valuation** v as follows:
 - For each proposition symbol p , if $p \in \Gamma_{\max}$ then $v(p) = T$, otherwise $v(p) = F$.
- Then argue that v **satisfies** Γ_{\max} using closure under derivability (using a soundness-like argument).
- Finally, v also satisfies Γ_0 , since $\Gamma_0 \subseteq \Gamma_{\max}$. So Γ_0 is satisfiable.