CS181u Applied Logic & Automated Reasoning

Lecture 4

Shannon Expansion
Adequate Sets of Connectives
Unit Propagation
DPLL
Binary Decision Diagrams

Variable Substitution (Replacement)

Variable replacement: given a Boolean formula f, a variable v, and an expression e, the notation $f[e/v]$ denotes the replacement of v with e in f.

Example: if $f = \neg x \wedge \neg y$ then

$$f[F/y] = \neg x \wedge \neg F = \neg x \wedge T = \neg x$$
$$f[T/x] = \neg T \wedge \neg y = F \wedge \neg y = F$$
$$f[\neg r/y] = \neg x \wedge \neg\neg r = \neg x \wedge r$$

Shannon Expansion

Given a variable x in a formula f, x can either be true or false. So, we can split the formula into two pieces—one in which x is asserted to be false and we plug F into f for x, and one in which x is aserted to be true and we plug T into f for x—and then take the disjunction.

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Hence, we can write the logical equivalence:

$$f \equiv (x = F) \wedge f[F/x] \quad \vee \quad (x = T) \wedge f[T/x]$$

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Hence, we can write the logical equivalence:

$$f \equiv (x = F) \wedge f[F/x] \quad \vee \quad (x = T) \wedge f[T/x]$$

Or, equivalently

$$f \equiv (\neg x) \wedge f[F/x] \quad \vee \quad x \wedge f[T/x]$$

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Shannon expand: $f \equiv \neg x \wedge f[F/x] \quad \vee \quad x \wedge f[T/x]$

$f[F/x_1]$ and $f[T/x_1]$ both have less than n variables and so can be written with just $\{\neg, \wedge, \vee\}$.

Boolean Satisfiability

DPLL uses **Unit Propagation**.

$$\phi = \{x \vee y, \neg x \vee z, z \vee w, x, y \vee v\}$$

A *unit clause* is a clause that is composed of a single literal, *u*.

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$\phi = \{z, x, y \vee v\}$

Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

Function : DPLL(ϕ)
Input : CNF formula ϕ over *n* variables
Output : true or false, the satisfiability of F
begin
  UnitPropagate(ϕ)
  if ϕ has false clause **then return** false
  if all clauses of ϕ satisfied **then return** true
  x $\leftarrow$ SelectBranchVariable(ϕ)
  return DPLL($\phi[x \mapsto \mathit{true}]$) $\vee$ DPLL($\phi[x \mapsto \mathit{false}]$)
end

DPLL Execution Example

$$\{z, x, y \vee v\}$$

DPLL Execution Example

$\{z, x, y \vee v\}$

$x \mapsto F$

UNSAT $\{z, F, y \vee v\}$

DPLL Execution Example

$$\begin{tikzcd}
& & \{z, x, y \vee v\} \arrow[dl, dash] \arrow[dr, dash] & \\
& x \mapsto F \arrow[dl, dash] & & x \mapsto T \arrow[d, dash] \\
\text{UNSAT} \quad \{z, F, y \vee v\} & & & \{z, T, y \vee v\}
\end{tikzcd}$$

DPLL Execution Example

$\{z, x, y \vee v\}$

$x \mapsto F$ $\quad$ $x \mapsto T$

UNSAT $\{z, F, y \vee v\}$ $\quad$ $\{z, T, y \vee v\}$

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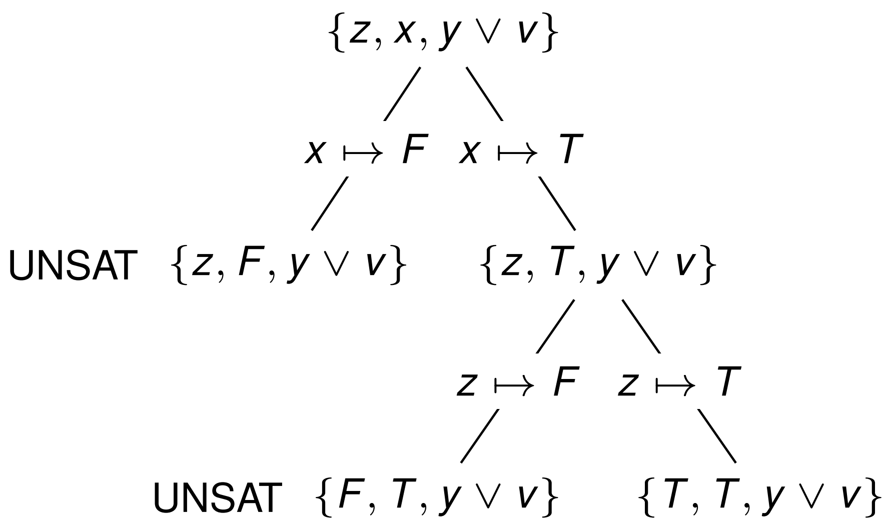
DPLL Execution Example



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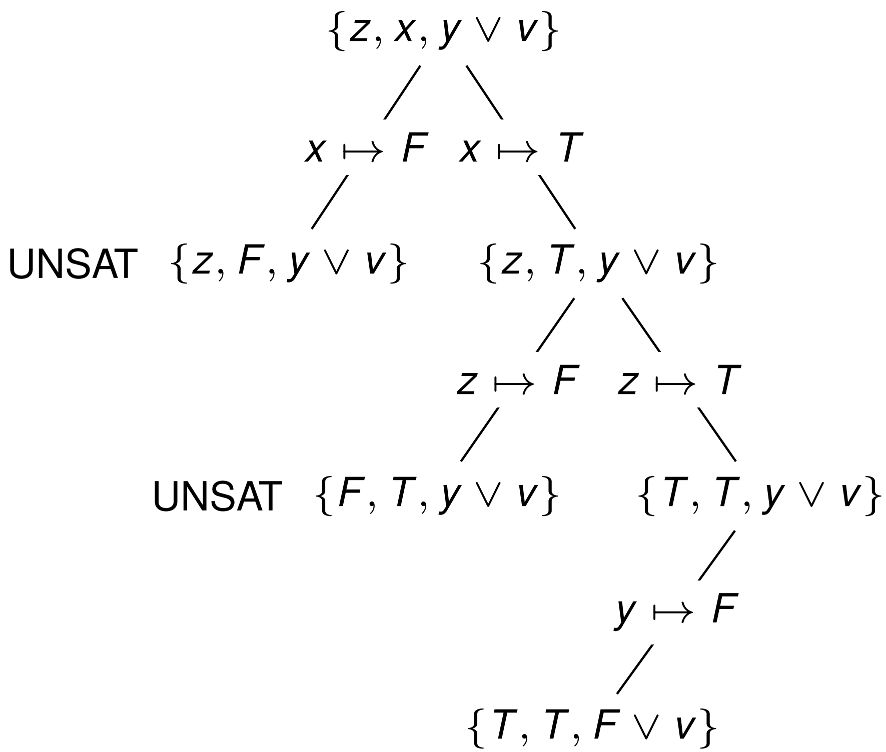

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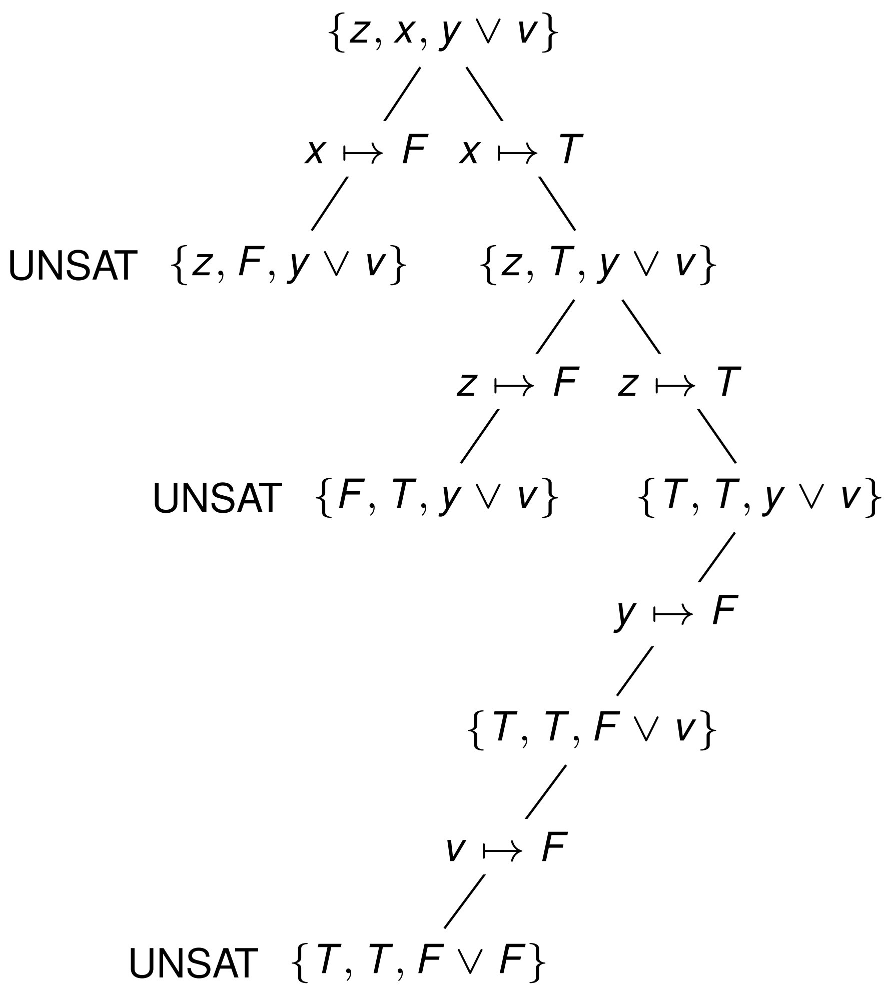

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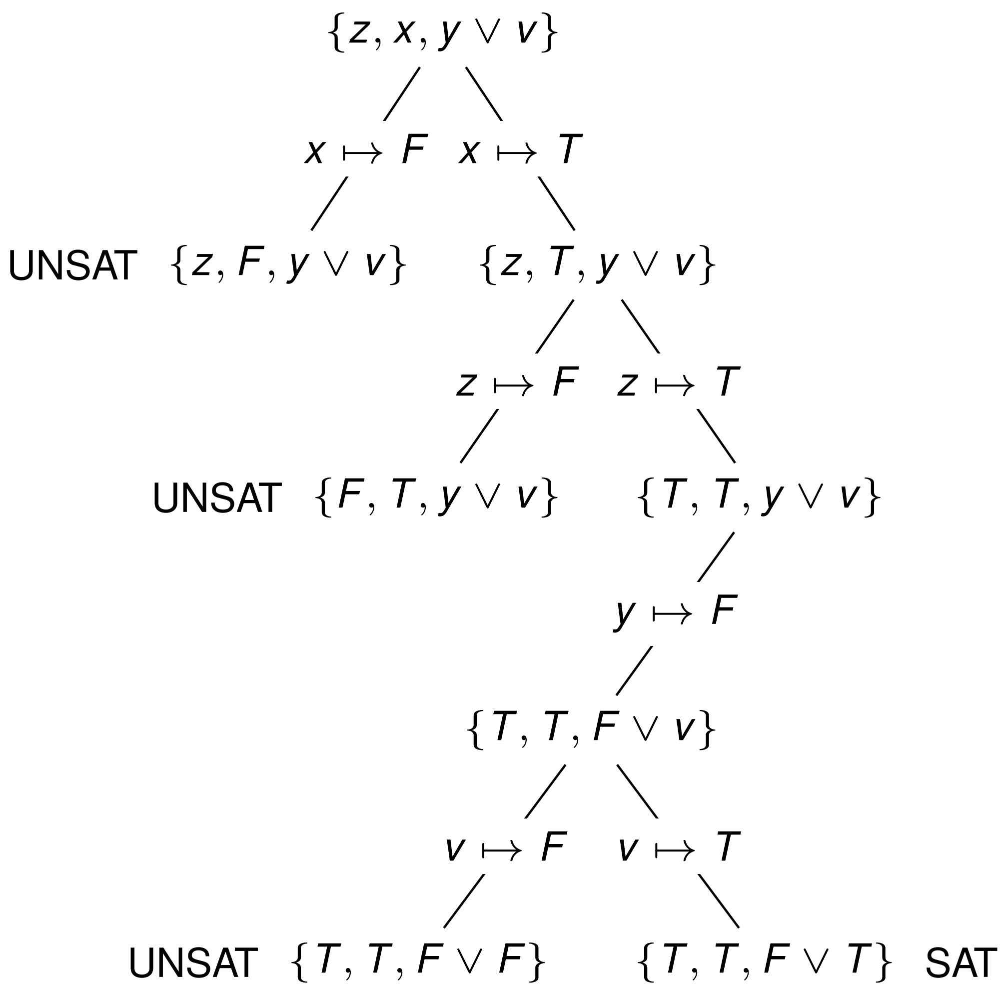


DPLL Execution Example

$$\{z, x, y \vee v\}$$

$$x \mapsto F \qquad x \mapsto T$$

UNSAT $\{z, F, y \vee v\}$ $\qquad$ $\{z, T, y \vee v\}$

$$z \mapsto F \qquad z \mapsto T$$

UNSAT $\{F, T, y \vee v\}$ $\qquad$ $\{T, T, y \vee v\}$

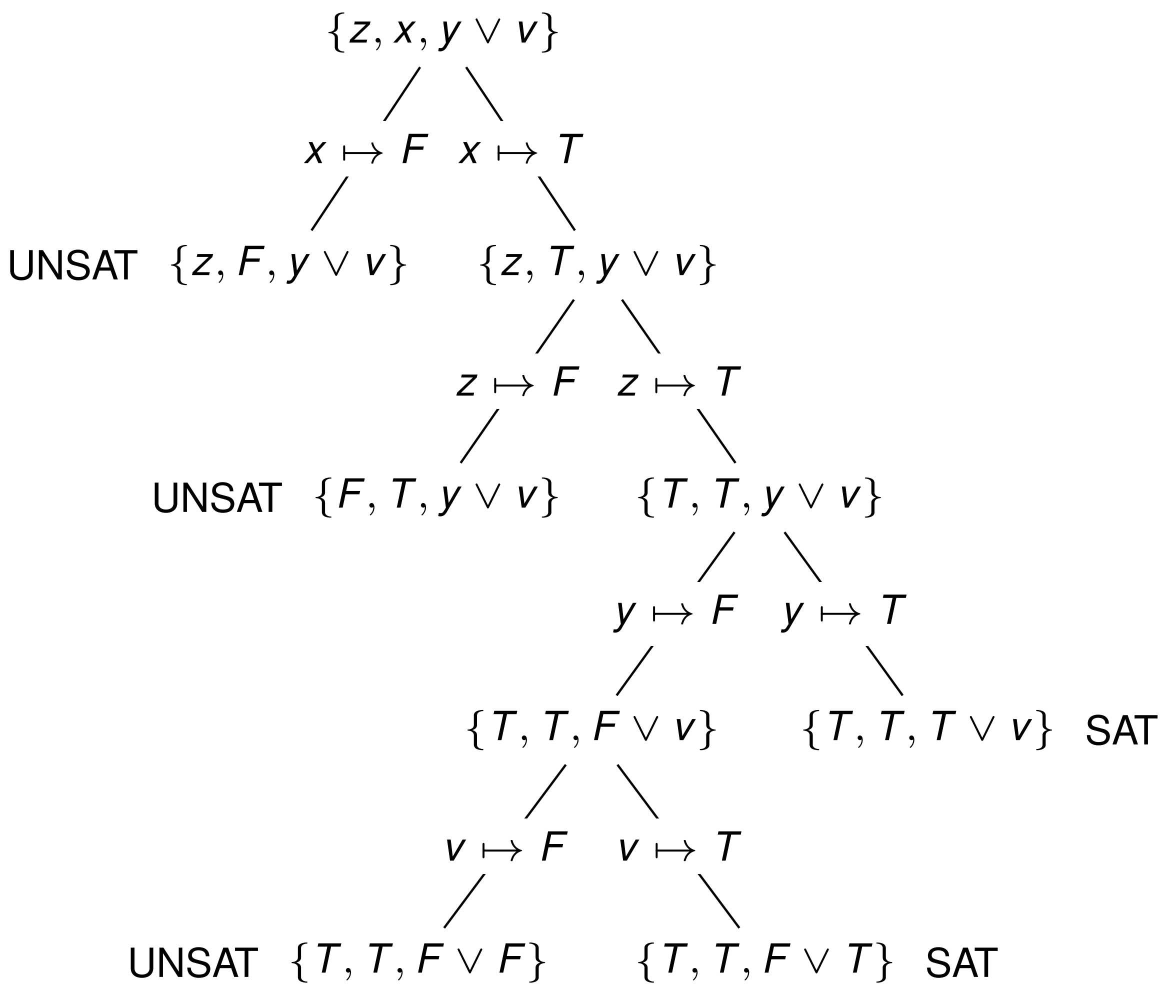
$$y \mapsto F \qquad y \mapsto T$$

$\{T, T, F \vee v\}$ $\qquad$ $\{T, T, T \vee v\}$ SAT

$$v \mapsto F \qquad v \mapsto T$$

UNSAT $\{T, T, F \vee F\}$ $\qquad$ $\{T, T, F \vee T\}$ SAT

DPLL Execution Example



Result: ϕ is satisfiable.

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   x $\leftarrow$ SelectBranchVariable(ϕ)
   return DPLL($\phi[x \mapsto \mathit{true}]$) $\vee$ DPLL($\phi[x \mapsto \mathit{false}]$)

end

Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

DPLL can be converted into a procedure for $\#$CNF-SAT.

Function : DPLL(ϕ, t)
Input : CNF formula ϕ over n variables; $t \in \mathbb{Z}$
Output : $\#\phi$, the model count of ϕ
begin
  UnitPropagate(ϕ)
  if ϕ has false clause **then return** 0
  if all clauses of ϕ satisfied **then return** 2^t
  x $\leftarrow$ SelectBranchVariable(ϕ)
  return DPLL($\phi[x \mapsto \mathit{true}], t - 1$) $+$ DPLL($\phi[x \mapsto \mathit{true}], t - 1$)
end

Counting with DPLL

$$\phi = \{x \vee y, \neg x \vee z, z \vee w, x, y \vee v\}, n = 5$$

$\{z, x, y \vee v\}$ $t = 5$

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$x \mapsto F$

0 $\{z, F, y \vee v\}$ $t = 4$

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$x \mapsto F$ $\qquad$ $x \mapsto T$

0 $\{z, F, y \vee v\}$ $t = 4$ $\qquad$ $\{z, T, y \vee v\}$ $t = 4$

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$x \mapsto F$ $\qquad$ $x \mapsto T$

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$z \mapsto F$

0 $\{F, T, y \vee v\}$ $t = 3$

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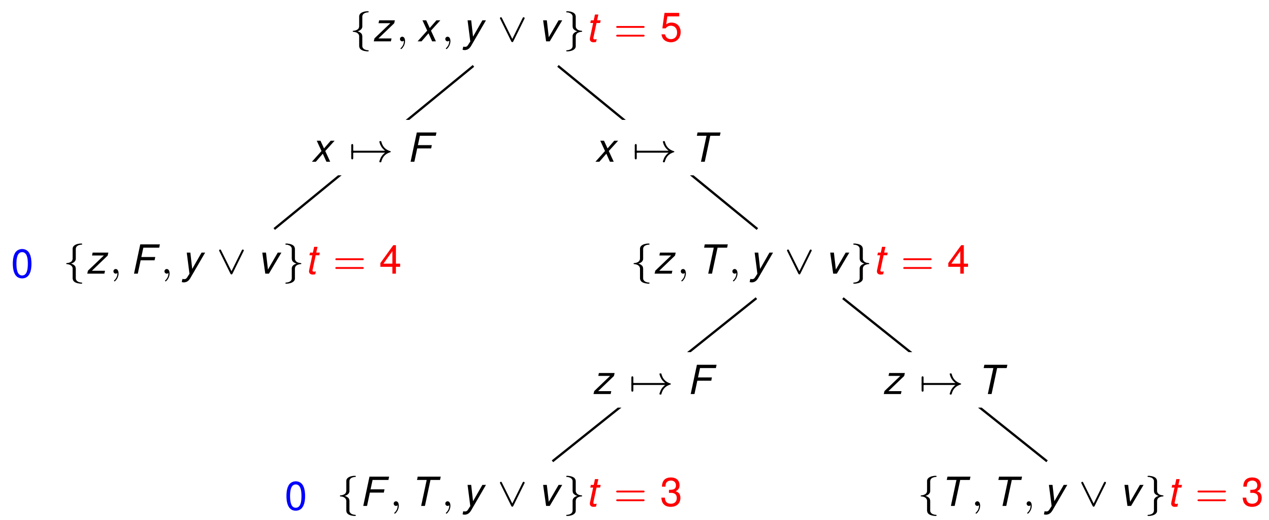

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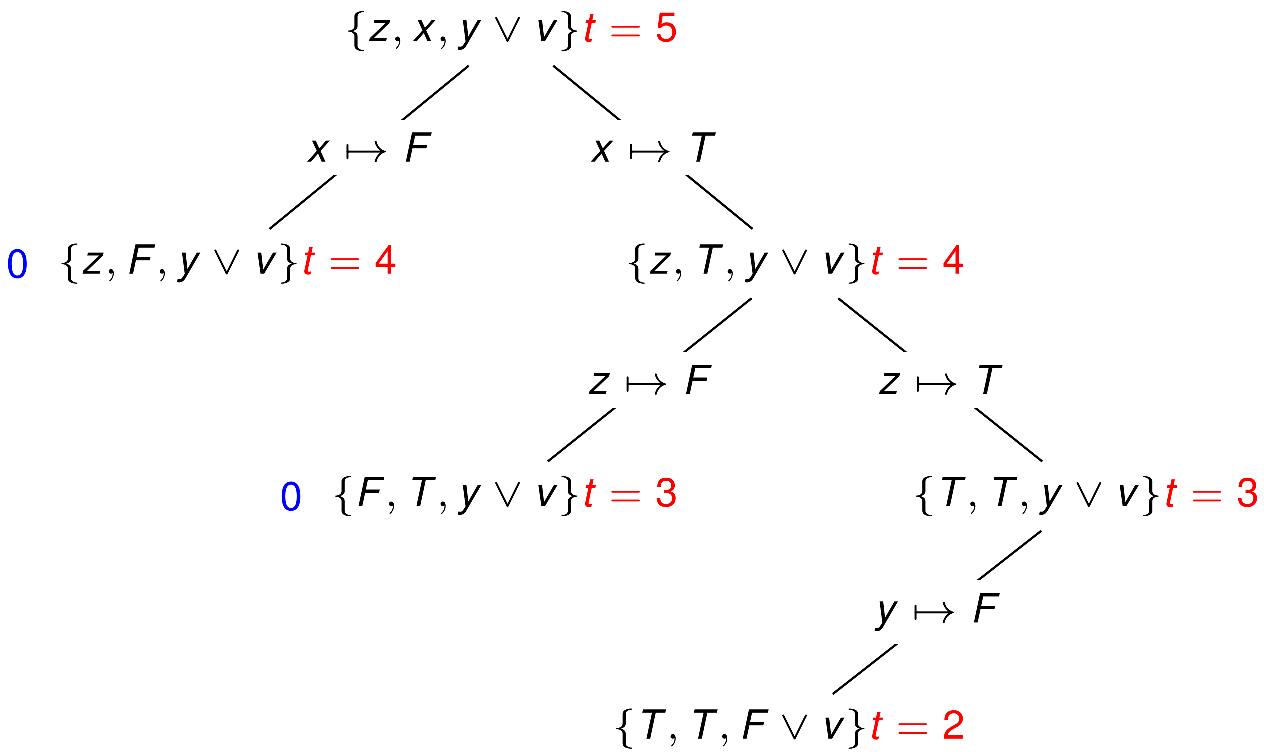

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- $\{z, x, y \vee v\}$ $t = 5$
  - $x \mapsto F$
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      - $z \mapsto T$
        - $\{T, T, y \vee v\}$ $t = 3$
          - $y \mapsto F$
            - $\{T, T, F \vee v\}$ $t = 2$
              - $v \mapsto F$
                - 0 $\{T, T, F \vee F\}$ $t = 1$

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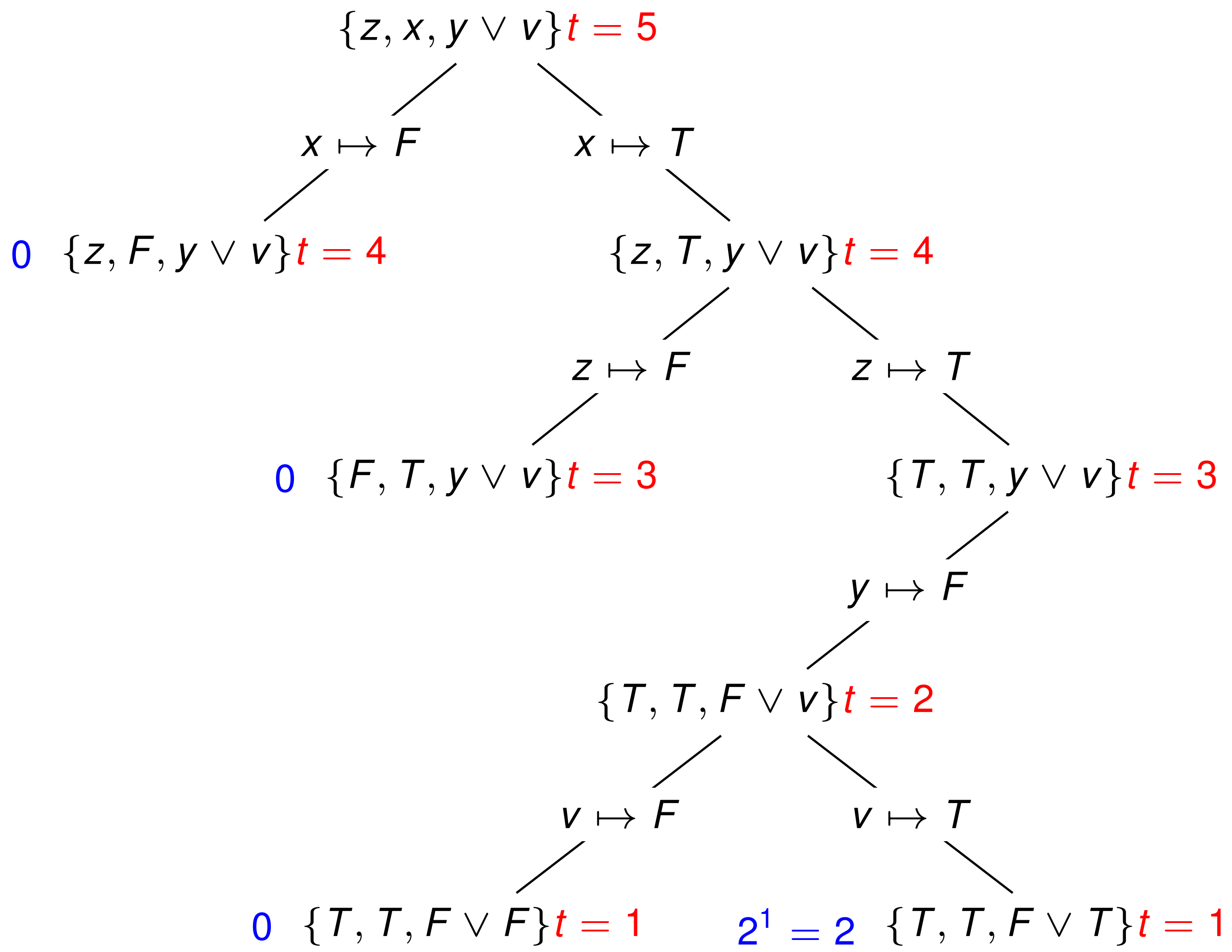

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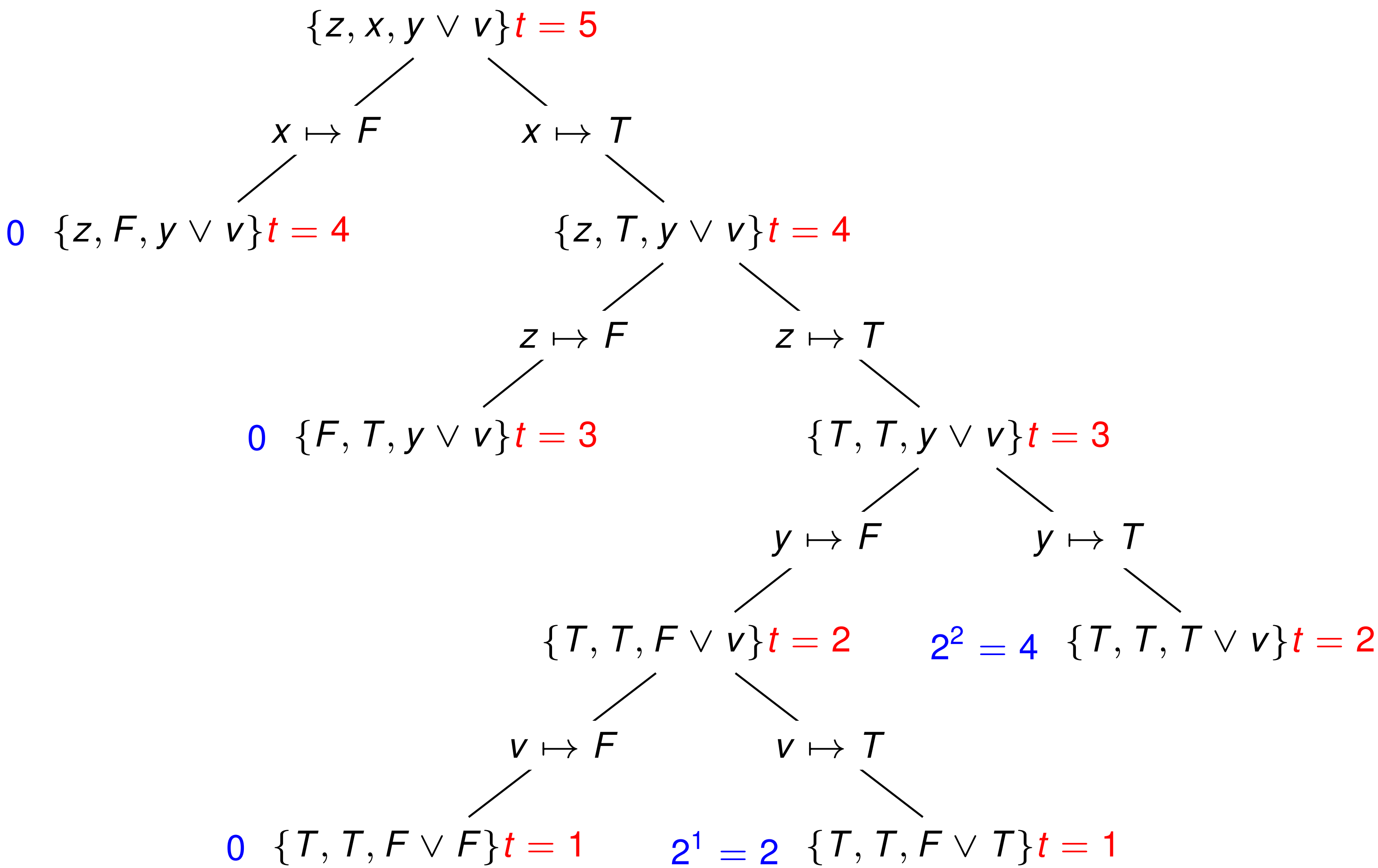

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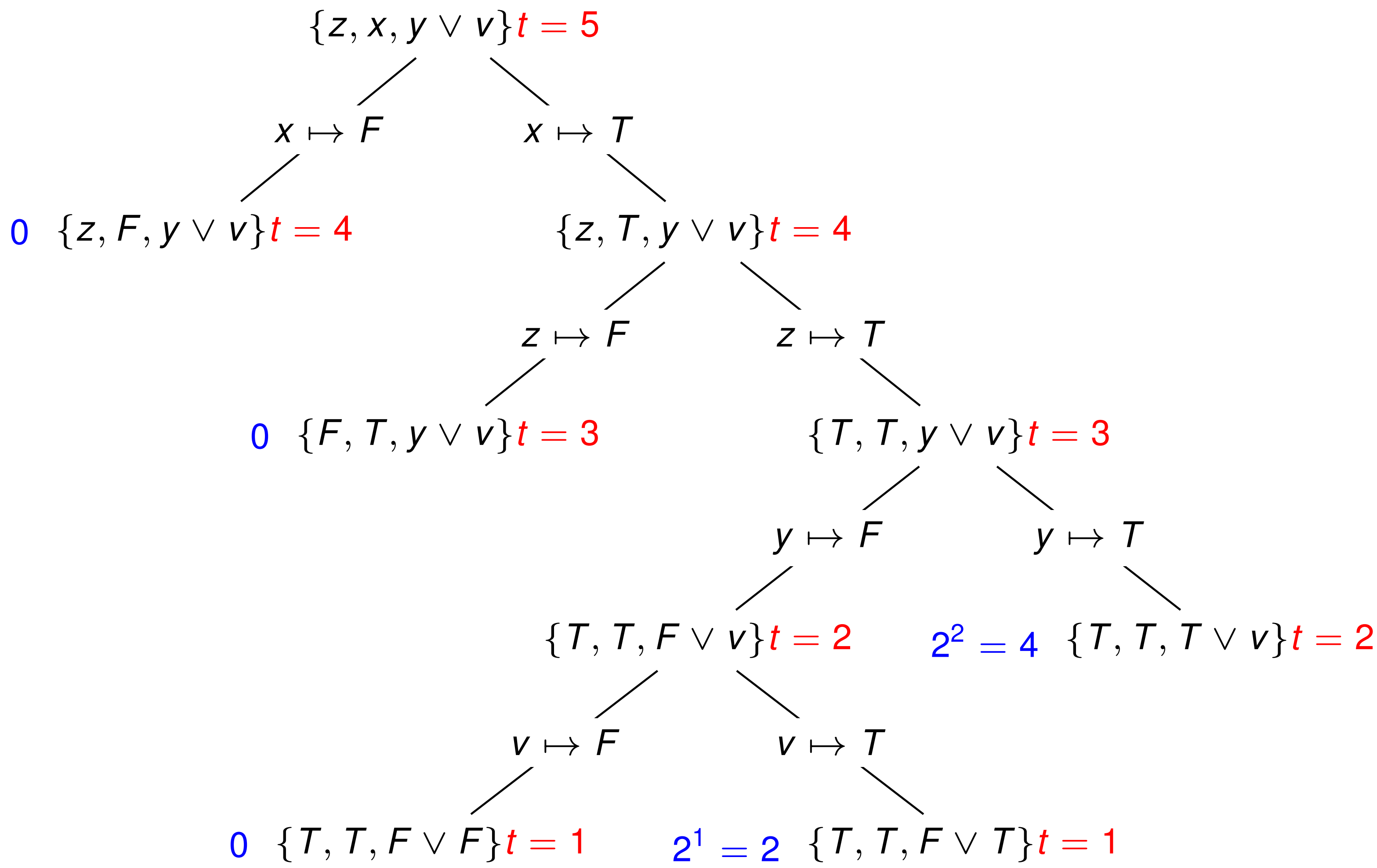


Result: $0 + 0 + 0 + 2 + 4 = 6$ models

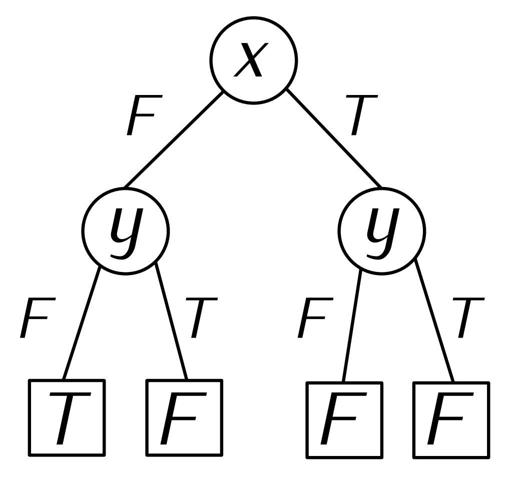
Binary Decision Diagrams

A **Binary Decision Diagram (BDD)** is a data structure for representing the truth values of formulas in propositional logic.

Example: consider the formula $\neg x \wedge \neg y$ and the truth table.

x	y	$\neg x \wedge \neg y$
F	F	T
F	T	F
T	F	F
T	T	F

The same information can be encoded in a decision tree.



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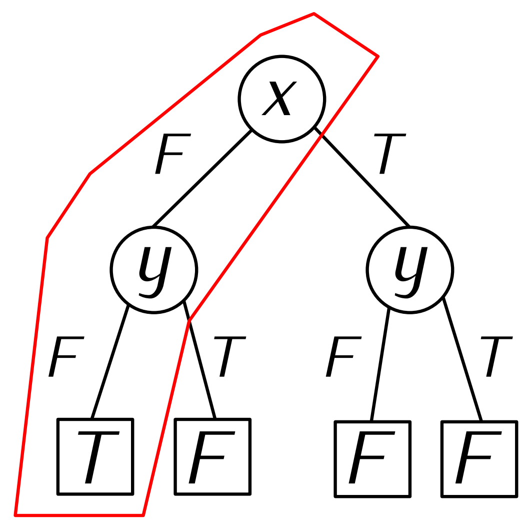

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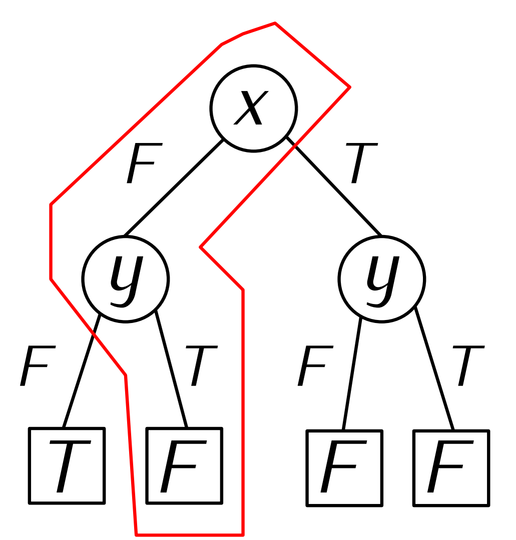

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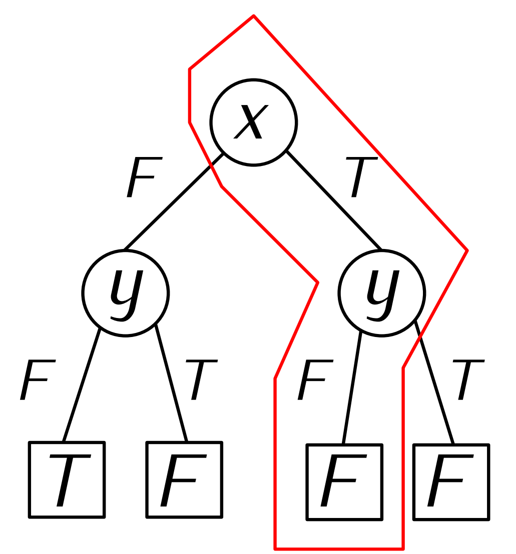

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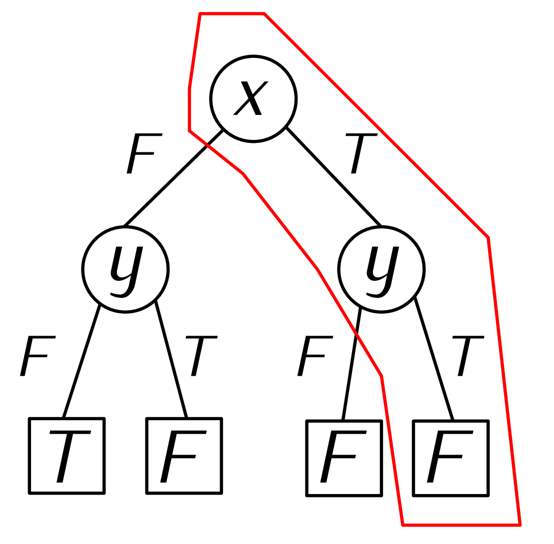
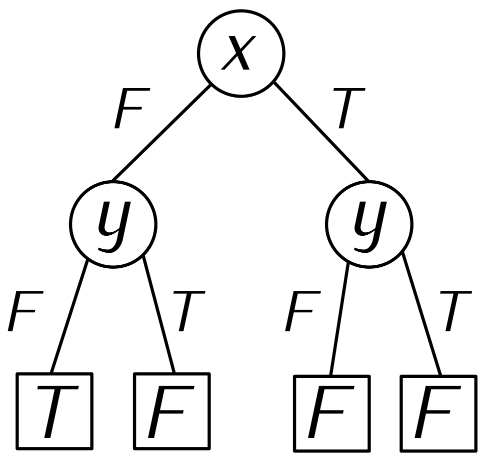


Binary Decision Diagrams

We can make the decision diagram more compact.



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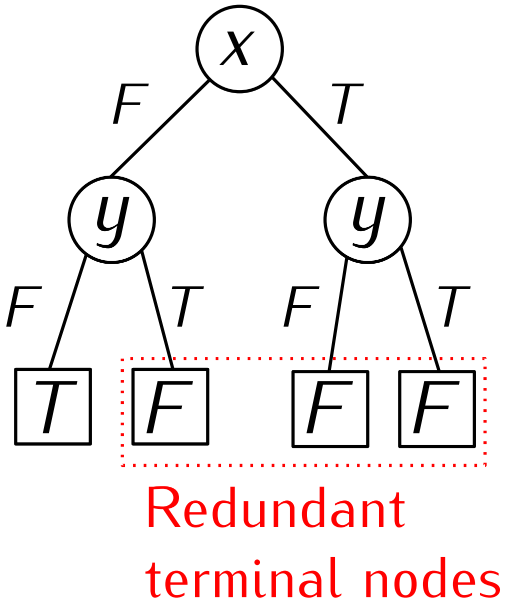

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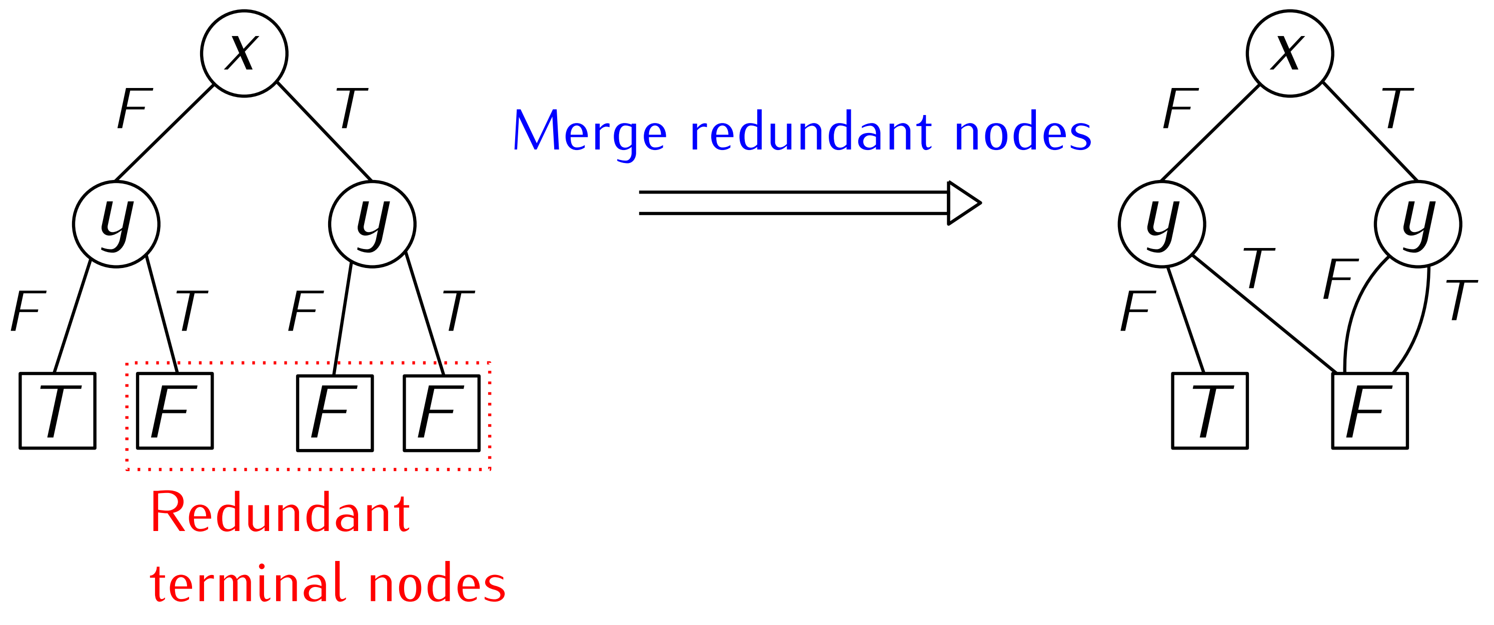

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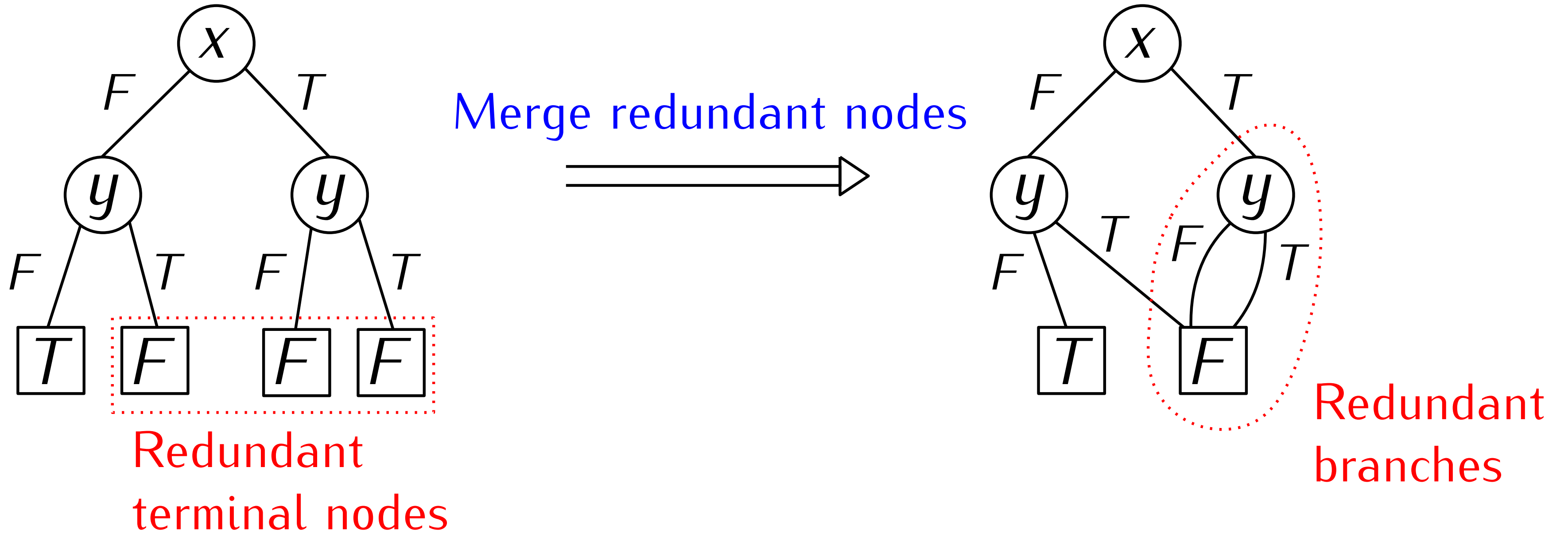

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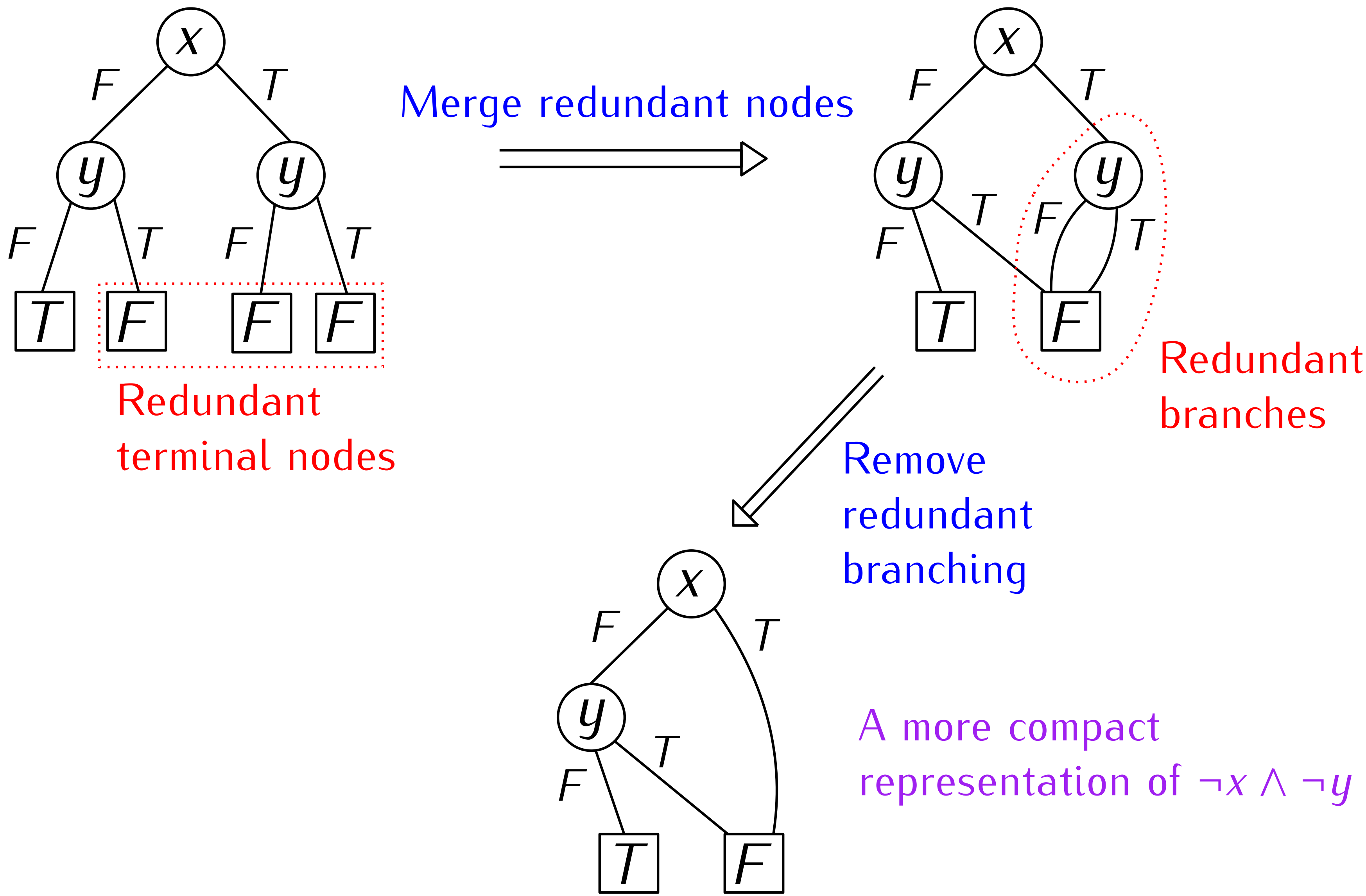
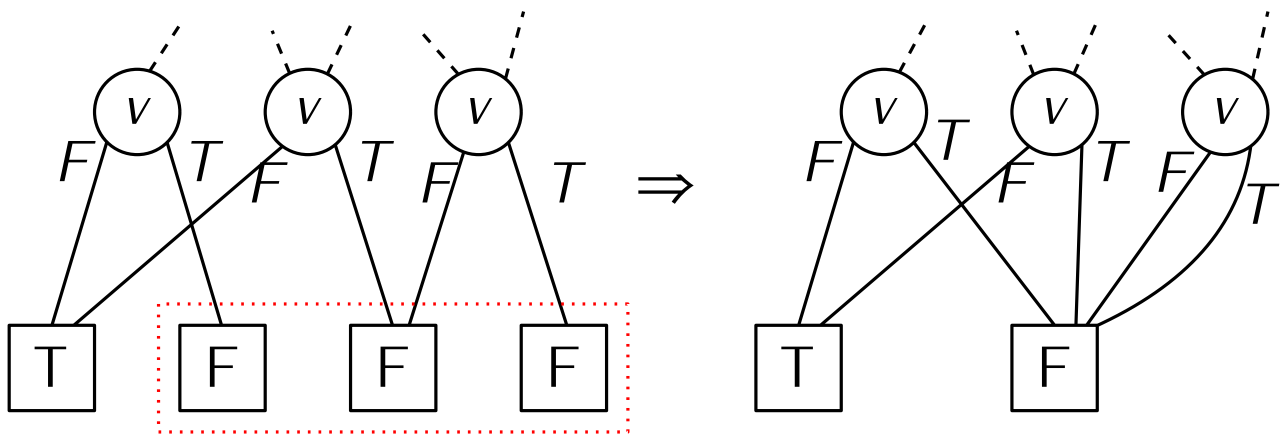
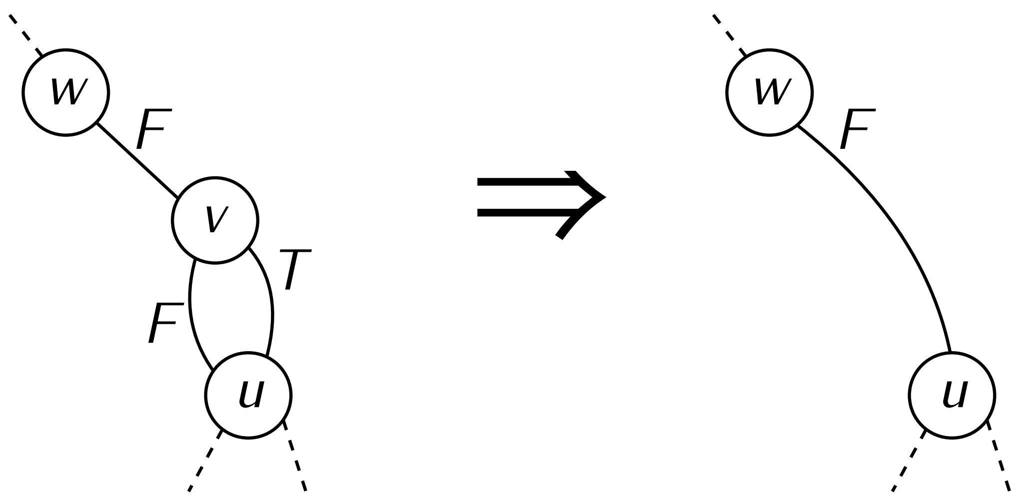


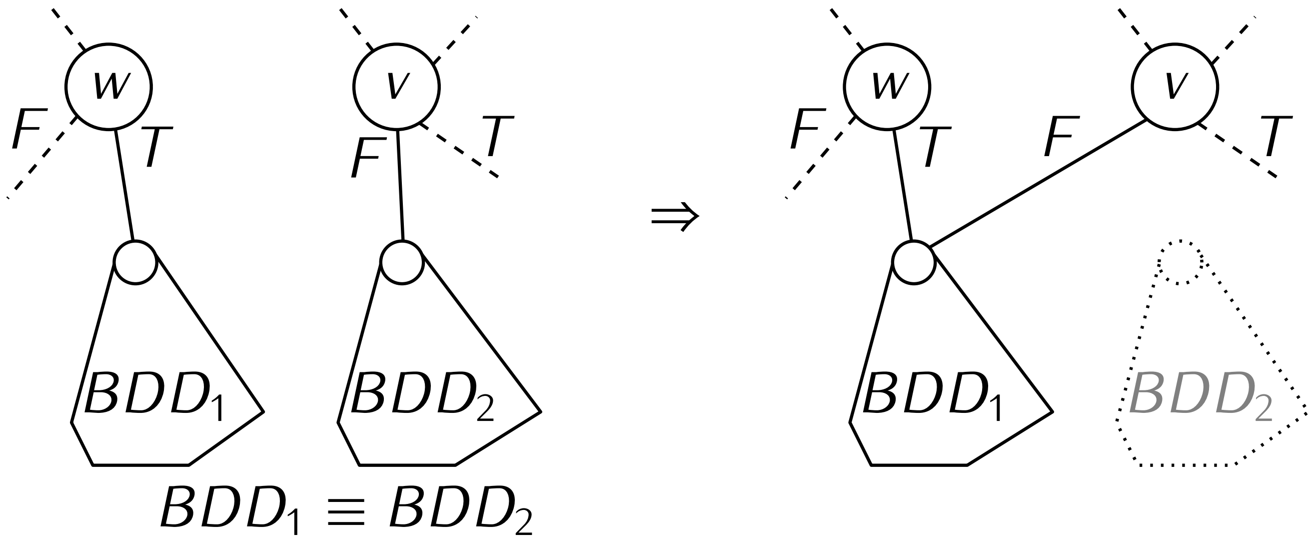
Reduction Rule 1: Merge duplicated terminal nodes.



Reduction Rule 2: Remove redundant tests.



Reduction Rule 3: Remove duplicate sub-BDDs.



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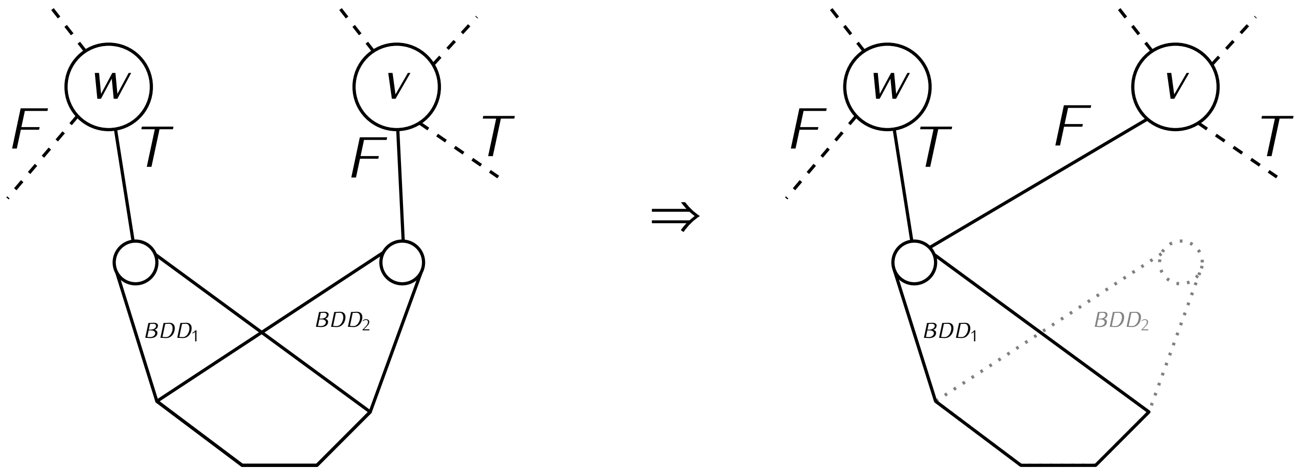
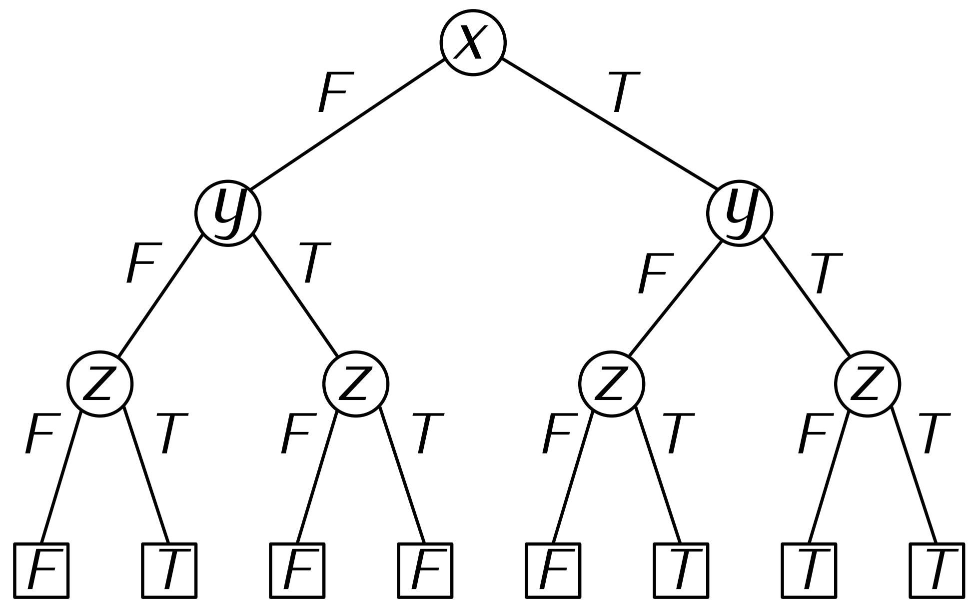


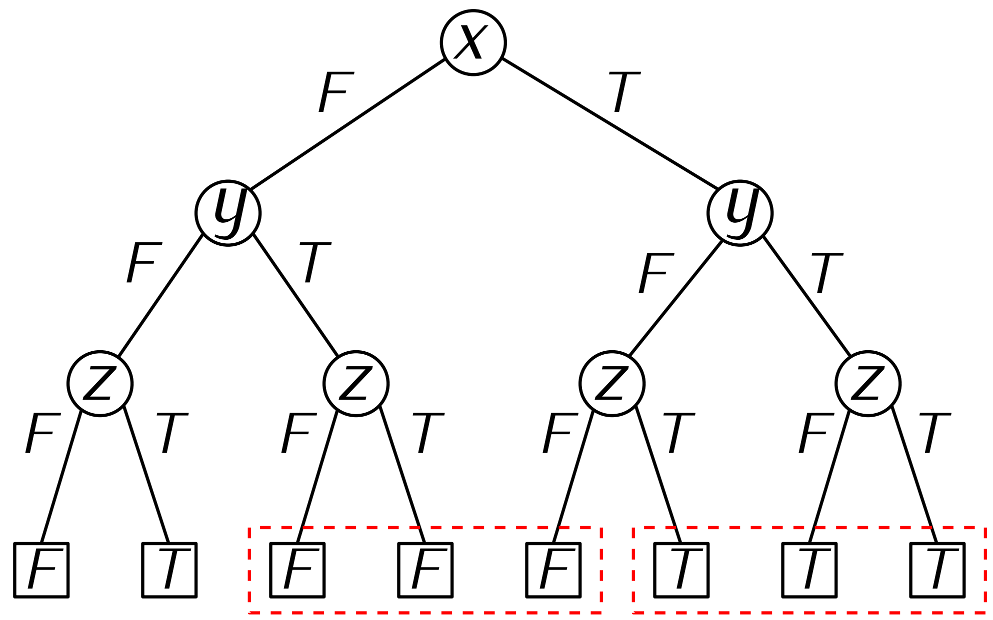
$BDD_1 \equiv BDD_2$

NOTE: They can be structurally identical, even if they overlap.

Example: Reduce the given BDD for the formula $(x \wedge y) \vee (\neg y \wedge z)$ to an ROBDD.



Example: Reduce the given BDD for the formula $(x \wedge y) \vee (\neg y \wedge z)$ to an ROBDD.



Merge duplicate terminals

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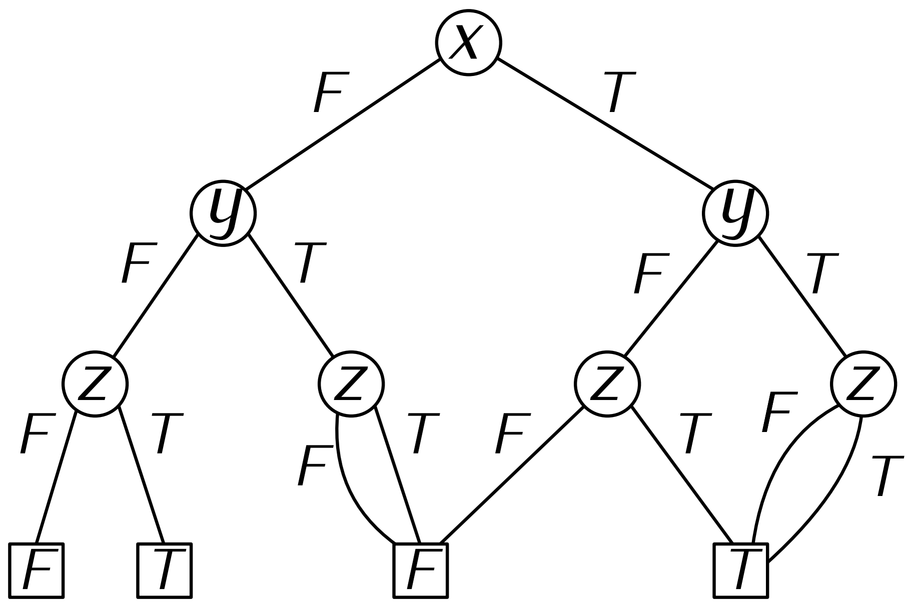

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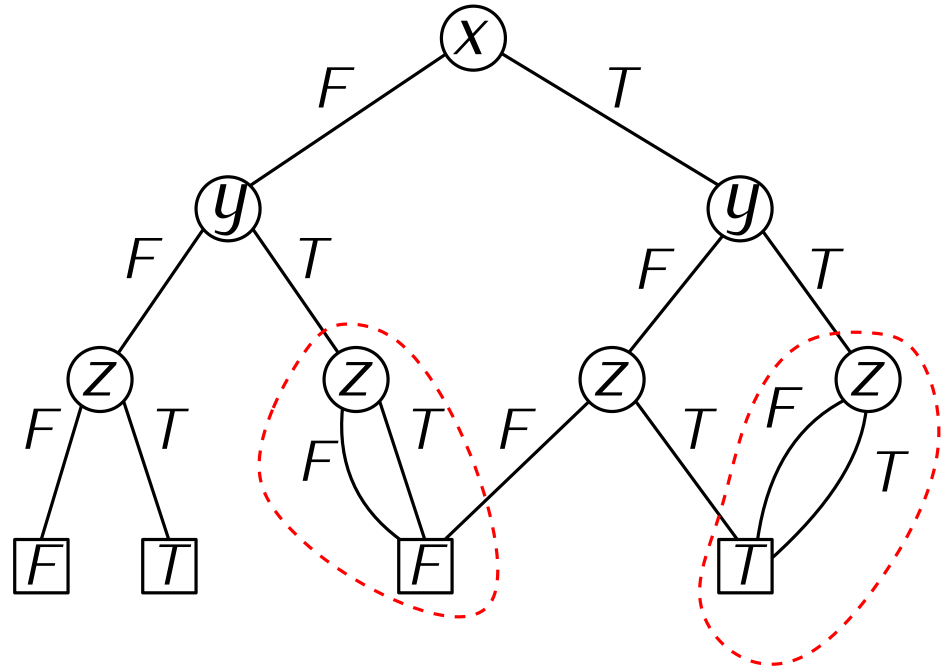

Delete redundant tests

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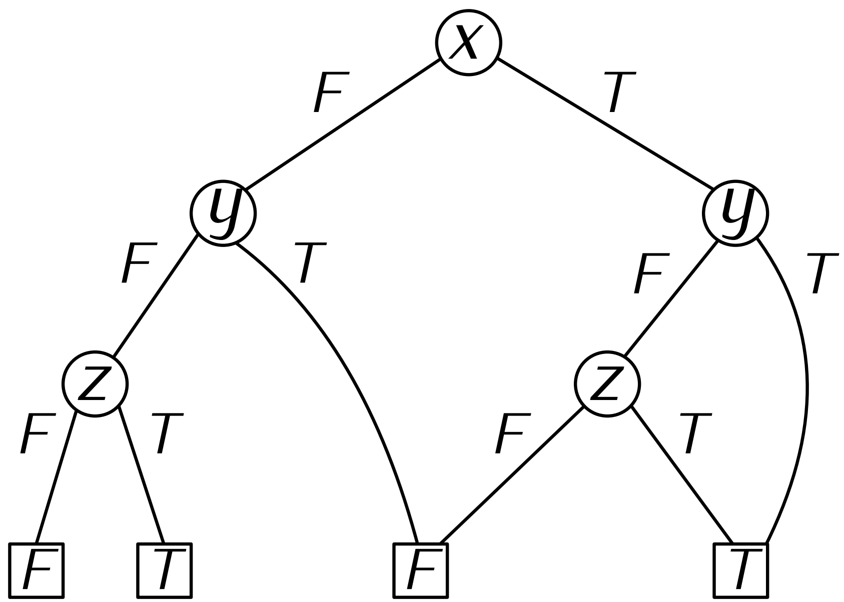

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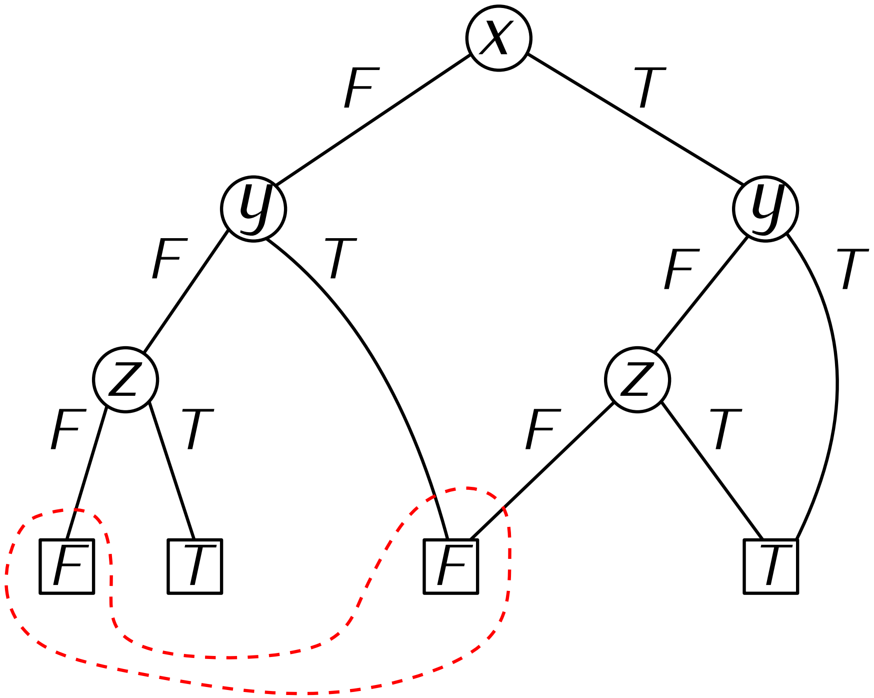

Merge more duplicate terminals

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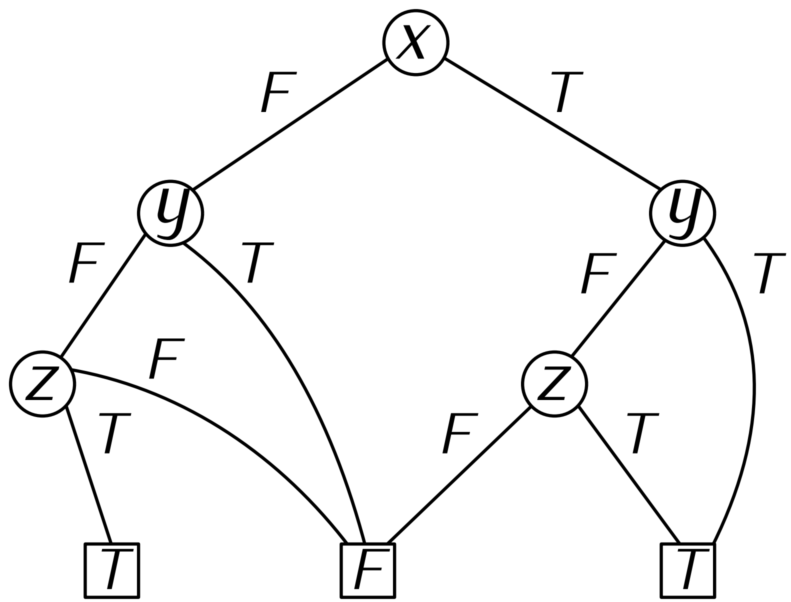

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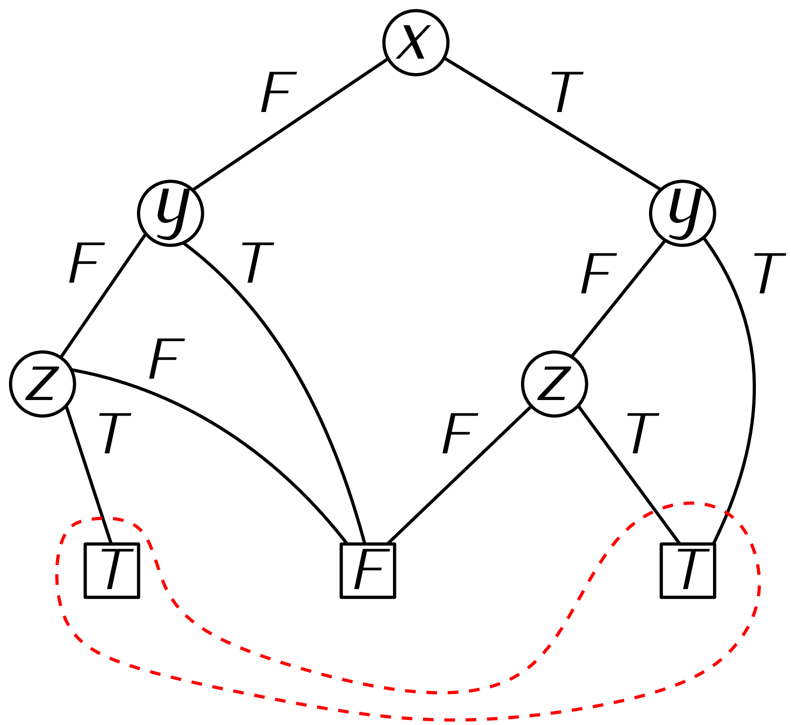

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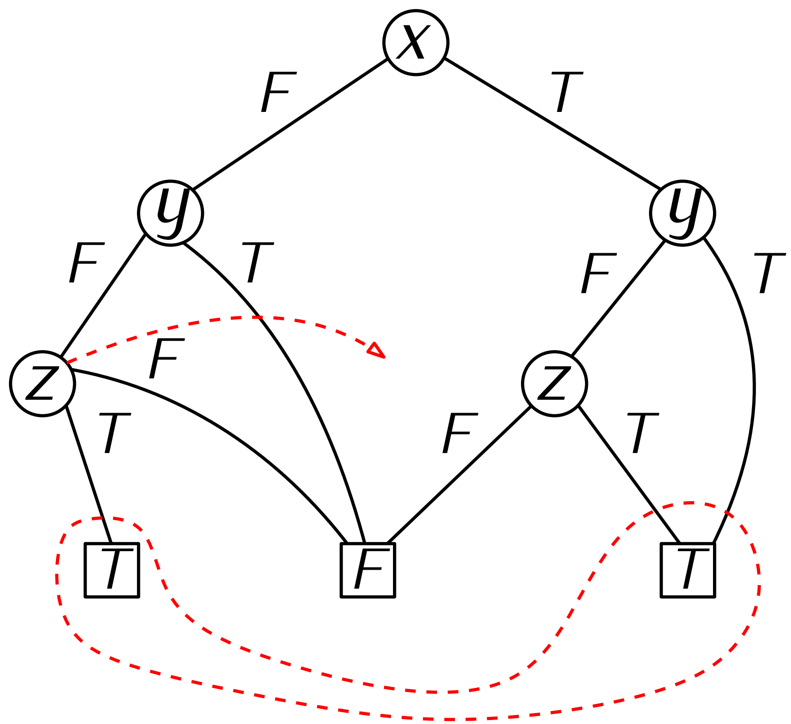

Redraw to "untangle"

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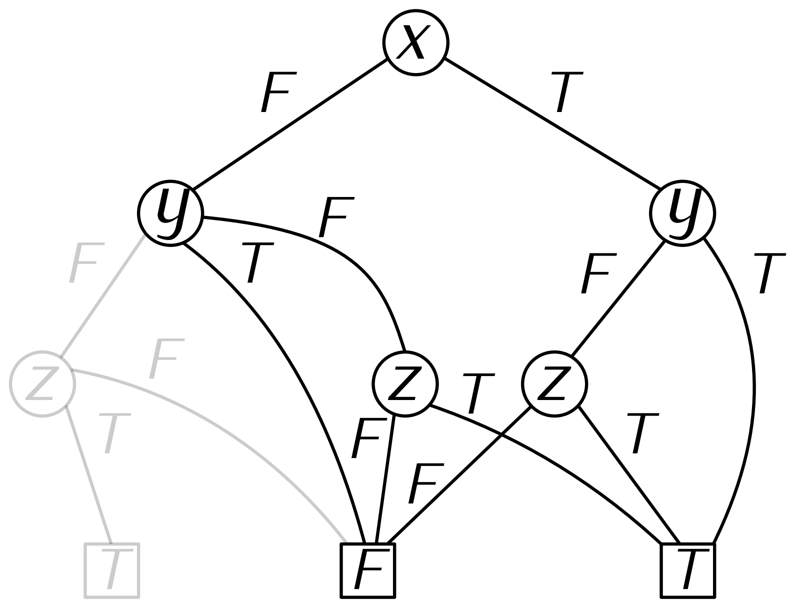

Redraw to “untangle”

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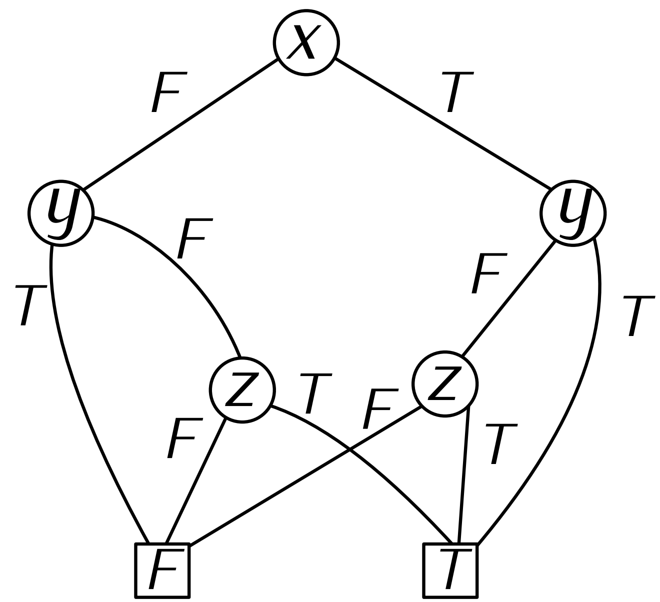

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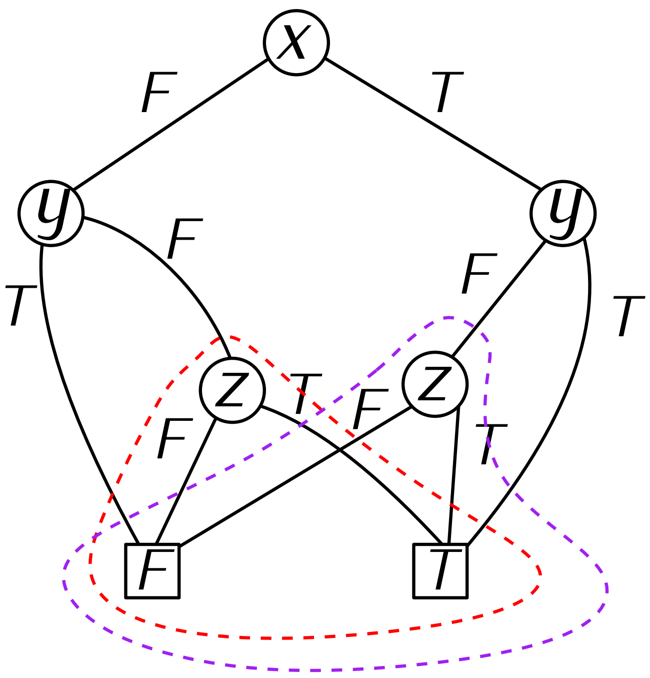

Remove duplicated sub-BDD

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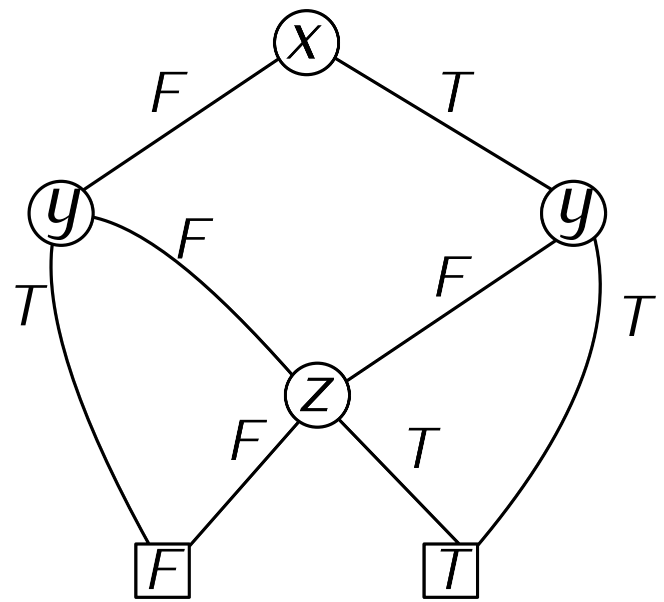

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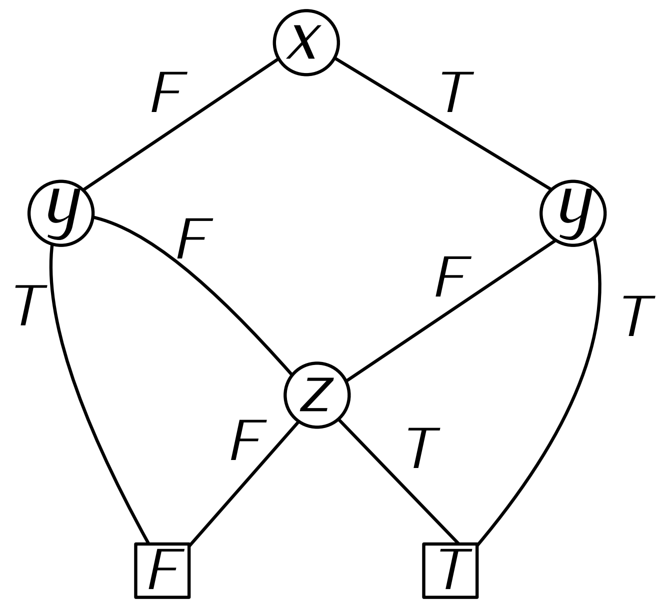

No more reduction possible

Important properties of BDDs

Ordered BDD (OBDD): variables are checked in a given order. E.g $x > y > z$.

Reduced OBDD (ROBDD): Cannot be reduced any further.

Theorem: ROBDDs are **unique** for a given ordering.

Important properties of BDDs

ROBDD **size** is sensitive to variable ordering!

Two different ROBDDs for the formula
x’ ⇔ x /\ y’ ⇔ y