CS181u Applied Logic
& Automated Reasoning

Lecture 6

Transition Systems
A transition system $\mathcal{M}$ can be specified by listing out all of the pieces.

States: $S = \{0, 1, 2, 3\}$

Initial States: $I = \{0\}$

Transitions:

$$R = \{(0, 1), (0, 2), (1, 3), (2, 3), (1, 0), (2, 0), (3, 1), (3, 2)\}$$
Symbolic Representation

Represent $M$ using Boolean logic.
Symbolic Representation

Represent $M$ using Boolean logic.

States

<table>
<thead>
<tr>
<th>States</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
</tbody>
</table>
Symbolic Representation

Represent $M$ using Boolean logic.

States | binary
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$y$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>
Symbolic Representation

Represent $M$ using Boolean logic.

<table>
<thead>
<tr>
<th>States</th>
<th>binary</th>
<th>truth values</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$x$</td>
<td>$y$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Symbolic Representation

Represent $M$ using Boolean logic.


Boolean state variables

$V = \{x, y\}$

<table>
<thead>
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<th></th>
<th>binary</th>
<th></th>
<th>truth values</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>0 0</td>
<td></td>
<td>F F</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>0 1</td>
<td></td>
<td>F T</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>1 0</td>
<td></td>
<td>T F</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>1 1</td>
<td></td>
<td>T T</td>
<td></td>
</tr>
</tbody>
</table>
Represent $M$ using Boolean logic.

\begin{align*}
\text{Boolean state variables} & \quad V = \{x, y\} \\
\end{align*}

<table>
<thead>
<tr>
<th>States</th>
<th>binary</th>
<th>truth values</th>
<th>Boolean formula</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$x$ $y$</td>
<td>$x$ $y$</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0 0</td>
<td>$F$ $F$</td>
<td>$\neg x \land \neg y$</td>
</tr>
<tr>
<td>1</td>
<td>0 1</td>
<td>$F$ $T$</td>
<td>$\neg x \land y$</td>
</tr>
<tr>
<td>2</td>
<td>1 0</td>
<td>$T$ $F$</td>
<td>$x \land \neg y$</td>
</tr>
<tr>
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<td>1 1</td>
<td>$T$ $T$</td>
<td>$x \land y$</td>
</tr>
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Symbolic Representation
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Represent $M$ using Boolean logic.
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Transitions:
Let the “next” state variables be $V' = \{x', y'\}$
Symbolic Representation

Represent $\mathcal{M}$ using Boolean logic.

**Transitions:**
Let the "next" state variables be $V' = \{x', y'\}$

$$R \equiv (x' = x \land y' = \neg y) \lor (x' = \neg x \land y' = y)$$
Represent $M$ using Boolean logic.

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“we can get from one state to the next by keeping one variable the same and negating the other”
Symbolic Representation

Represent $M$ using Boolean logic.

Transitions:
Let the “next” state variables be $V' = \{x', y'\}$

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Explicit transitions:

- $(0, 1)$
- $(2, 3)$
- $(1, 3)$
- $(0, 2)$
- $(1, 0)$
- $(3, 2)$
- $(3, 1)$
- $(2, 0)$

“we can get from one state to the next by keeping one variable the same and negating the other”
Symbolic Representation

\[ R \equiv (x' = x \land y' = \neg y) \lor (x' = \neg x \land y' = y) \]

BDD for R
Symbolic Model Checking: \(10^{20}\) States and Beyond

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School of Computer Science
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Abstract

Many different methods have been devised for automatically verifying finite state systems by examining state-graph models of system behavior. These methods all depend on decision procedures that explicitly represent the state space using a list or a table that grows in proportion to the number of states. We describe a general method that represents the state space symbolically instead of explicitly. The generality of our method comes from using a dialect of the Mu-Calculus as the primary specification language. We describe a model checking algorithm for Mu-Calculus formulas that uses Bryant’s Binary Decision Diagrams (1986) to represent relations and formulas. We then show how our new Mu-Calculus model checking algorithm can be used to derive efficient decision procedures for CTL model checking, satisfiability of linear-time temporal logic formulas, strong and weak observational equivalence of finite transition systems, and language containment for finite \(\omega\)-automata. The fixed point computations for each decision procedure are sometimes complex, but can be concisely expressed in the Mu-Calculus. We illustrate the practicality of our approach to symbolic model checking by discussing how it can be used to verify a simple synchronous pipeline circuit.
This phase: model checking

This phase will focus mostly on model checking.

Proof-based systems are good for programs that take input and then compute a result.

Model checking is good for programs that define reactive systems.

A reactive system consists of multiple components operating concurrently and indefinitely.
Next Few Weeks:

Linear Temporal Logic (LTL)

We will assign symbols for expressing temporal system requirements like *always* \((G)\), *eventually* \((F)\), *next* \((X)\), *until* \((U)\), and a few more. We will give a formal and unambiguous semantics to these symbols.

Transition Systems

We will learn a formal system of specifying transition systems (which we often depict as a transition diagram).

Concurrency Concepts

Safety, liveness, mutual exclusion, …

Temporal Logic Software

Symbolic Model Verifier (NuSMV)
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Symbolic Model Verifier (NuSMV)
2.5 Bibliographic Notes

Transition systems. Keller was one of the first researchers that explicitly used transition systems [236] for the verification of concurrent programs. Transition systems are used
Reactive System Code
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Requirements
Goal: show that

Reactive System Code satisfies Requirements
Goal: show that

Reactive System Code satisfies $\models$ Requirements
Goal: show that

Reactive System Code satisfies

\[ \phi \]

Requirements

Transition System

Temporal Logic Formula \( \phi \)
Goal: show that

\[ \text{Reactive System Code} \ satisfies \ \models \ \text{Requirements} \]

\[ \text{Transition System} \ satisfies \ \models \ \text{Temporal Logic Formula} \ \phi \]
Goal: show that

Reactive System Code satisfies $\models$ Requirements

Transition System satisfies $\models$ Temporal Logic Formula $\phi$

Model Checking
Case Study: Mutual Exclusion

Motivation: reactive system is a bank, two ATMs, and two customers that share an account.
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Current balance $b = 1000$
$C_1$ wants to deposit $d_1 = 100$
$C_2$ wants to deposit $d_2 = 100$
Case Study: Mutual Exclusion

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ATM$_1$ writes $b = b_1 + d_1 = 1100$

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ATM$_1$ writes $b = b_1 + d_1 = 1100$
ATM$_2$ writes $b = b_2 + d_2 = 1100$
Final balance $b = 1100$
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ATM$_1$ writes $b = b_1 + d_1 = 1100$
ATM$_2$ writes $b = b_2 + d_2 = 1100$
Final balance $b = 1100$

One ATM shouldn’t read the balance while another is performing a transaction! Race condition.
while (true) {

  // non-critical code
  printWelcomeMessage();

  // critical section code
  updateBankBalance();

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    printWelcomeMessage();

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}
We want to focus on the mutual exclusion logic.
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**Abstraction:** forget about details we don’t care about at the moment.
while (true) {
    [
        [[extra code to help with mutual exclusion]]
    
    // non-critical code
    printWelcomeMessage();

    [
        [[extra code to help with mutual exclusion]]
    
    // critical section code
    updateBankBalance();

    [
        [[extra code to help with mutual exclusion]]
    }
}

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**Abstraction**: forget about details we don’t care about at the moment.
while (true) {
  [[non-critical mutual exclusion code]]

  [[waiting to enter critical section]]

  [[critical section mutual exclusion code]]
}
Define a process, proc, with an id, partial access to the other process, and a shared variable representing whose turn it is.

Labels: n (non-critical), w (waiting), c (critical section).
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Labels: \texttt{n} (non-critical), \texttt{w} (waiting), \texttt{c} (critical section).

\begin{verbatim}
proc(id, other, turn)
  while(true)
    n: b := TRUE; turn = (id + 1) % 2;
    w: wait until (!other.b | turn = id)
    c: b := FALSE;
}
\end{verbatim}

\[ P_0 = \text{proc}(0, P_1, 0) \]
\[ P_1 = \text{proc}(1, P_0, 0) \]
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Labels: n (non-critical), w (waiting), c (critical section).

\[
\begin{align*}
\text{proc(} & \text{id, other, turn) } \\
\text{while(} & \text{true})\{ \\
\text{n:} & \quad b := \text{TRUE}; \text{ turn} = (\text{id} + 1) \text{ mod } 2; \\
\text{w:} & \quad \text{wait until (! other.b | turn = id) } \\
\text{c:} & \quad b := \text{FALSE}; \\
\} \\
\end{align*}
\]

\[
\begin{align*}
P_0 & = \text{proc(0, P_1, 0)} \\
P_1 & = \text{proc(1, P_0, 0)} \\
\end{align*}
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\(P_0 || P_1\) is a simple reactive system.

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}
\end{verbatim}

\begin{align*}
P_0 &= \texttt{proc(0, } P_1, 0) \\
P_1 &= \texttt{proc(1, } P_0, 0)
\end{align*}

\textit{P}_0 || \textit{P}_1 \text{ is a simple reactive system.}

\textbf{Mutual Exclusion Requirement:} \textit{P}_0 \text{ and } \textit{P}_1 \text{ are never both in the critical section (at line } \textit{c}) \text{ at the same time.}

\textbf{Idea:} Variable \textit{turn} keeps track of whose turn it is. Variable \textit{b} is only \textsc{true} if about to execute line \texttt{w} or \texttt{c}.
\begin{verbatim}
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The \textbf{state} of a single process consists of the values of all variables relevant to that process.

There is an extra “variable”, the \textbf{program counter}, typically called \( pc_i \), which keeps track of which line of code is \textbf{about} to execute.
while (true)
{
    n: b := TRUE; turn = (id + 1) % 2;
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}

\[ pc_i = n \]

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\[ P_0 = \text{proc}(0, P_1, 0) \]
\[ P_1 = \text{proc}(1, P_0, 0) \]

The state of the combined system, \( P_0 \parallel P_1 \) is given by the values of all variables together. The state of this system is completely determined by the tuple

\[ (pc_0, pc_1, turn, b_0, b_1) \]
In class activity:

Building intuition for transition systems using mutual exclusion as a case study.
Transition system for $P_0||P_1$ from in-class activity.
Transition Systems

A transition system $\mathcal{M} = (S, I, \rightarrow, L)$ is a set of states $S$ and a set of initial states $I$, along with a transition relation $\rightarrow$ and labelling function $L$.

The transition relation $\rightarrow$ is equivalent to a set of directed graph edges, with the states as nodes.

For example, $((n_0, n_1, 0, F, F), (n_0, w_1, 0, F, T)) \in \rightarrow$

Alternatively, we can write $(n_0, n_1, 0, F, F) \rightarrow (n_0, w_1, 0, F, T)$.

Important assumption: no dead states. Every state has an outgoing transition, even if only to itself.
Transition Systems, execution paths

A path in a transition system $\mathcal{M} = (S, I, \rightarrow, L)$ is an infinite sequence of states $s_1, s_2, s_3, \ldots$ such that $s_1 \in I$ and for every $i \geq 1$, $s_i \rightarrow s_{i+1}$.
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For example, one path from our two-process mutual exclusion transition diagram:

$((n_0, n_1, 0, F, F), (n_0, w_1, 0, F, T), (n_0, c_1, 0, F, T))^\omega$
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For example, one path from our two-process mutual exclusion transition diagram:

$$((n_0, n_1, 0, F, F), (n_0, w_1, 0, F, T), (n_0, c_1, 0, F, T))^\omega$$

We will use the symbol $\pi$ for paths. We write $\pi = s_1, s_2, s_3 \ldots$. We write $\pi^i$ to indicate the $i$th suffix of $\pi$. E.g. $\pi^3 = s_3, s_4, s_5 \ldots$
Transition System Example

\[ S = \{0, 1, 2\} \quad I = \{0\} \quad AP = \{p, q, r\} \]
\[ \rightarrow = \{(0, 1), (1, 0), (0, 2), (1, 2)\} \]
\[ L(0) = \{p, q\} \quad L(1) = \{q, r\} \quad L(2) = \{r\} \]
Transition System Example

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Remember the big picture

Reactive System Code \(\models\) Requirements

Transition System \(\models\) Temporal Logic Formula \(\phi\)

Model Checking