Automatically Computing Path Complexity of Programs

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Department of Computer Science
University of California, Santa Barbara

ESEC FSE 2015

Overview: What did we do?

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PAth
Complexity
Analyzer
(PAC)

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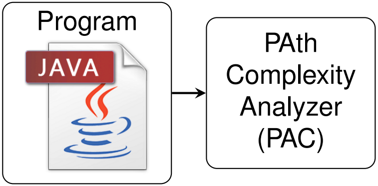

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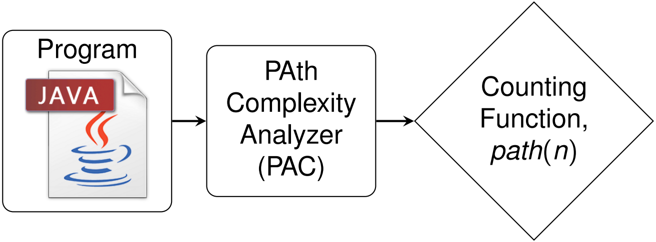

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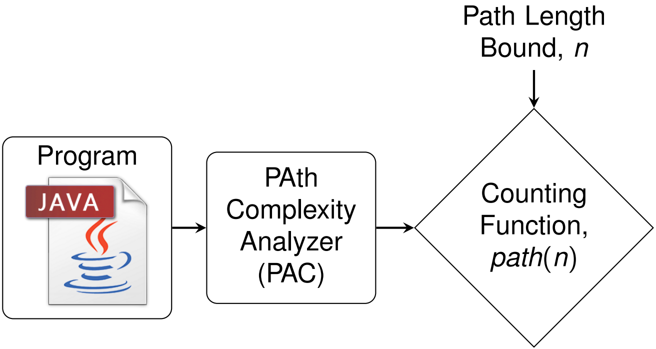

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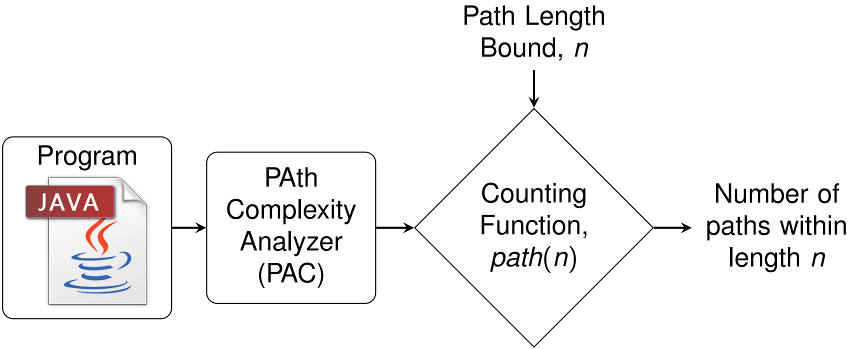

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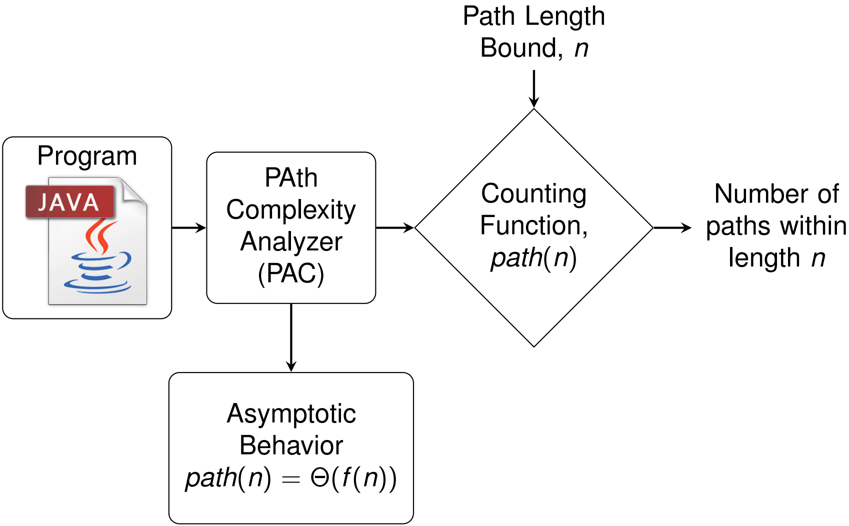

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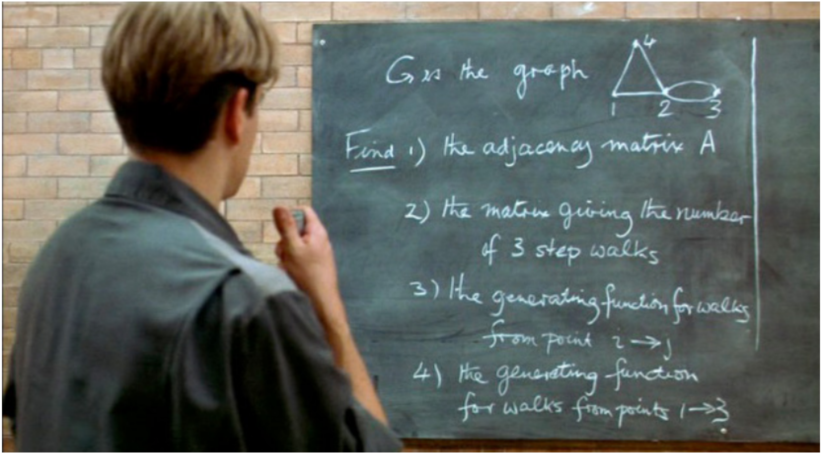

Outline

Motivation

Path Complexity

Experiments

Motivation

Program Path Coverage

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- Practical solution: explore up to a given depth bound.
- We propose a metric, the **path complexity**, an upper bound on the number of paths needed to explore up to a given depth.

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Program Path Coverage

- Modern automated software testing techniques focus on program path coverage.
- The number of execution paths could be infinite.
- Practical solution: explore up to a given depth bound.
- We propose a metric, the **path complexity**, an upper bound on the number of paths needed to explore up to a given depth.
- This provides a measure of the difficulty of achieving path coverage.

Path Complexity

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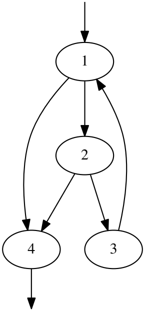

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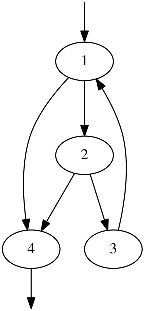

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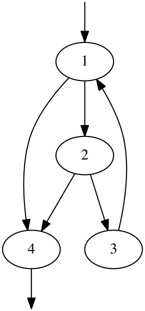

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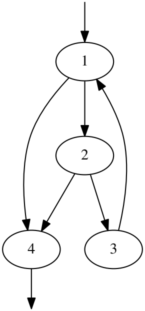

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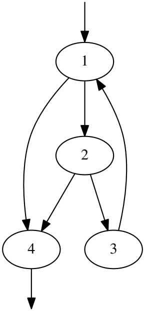

n	*count*(*n*)	*path*(*n*)
0	0	0
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2	1	2
3	0	2
4	1	3
5	1	4
6	0	4
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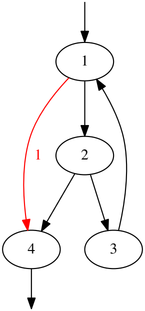

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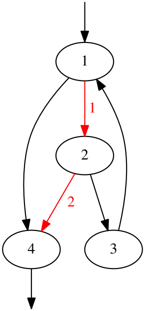

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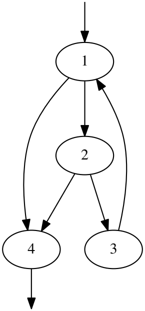

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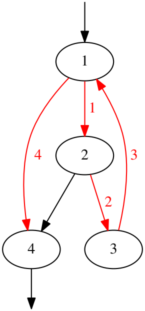

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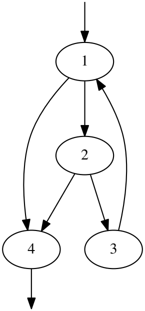

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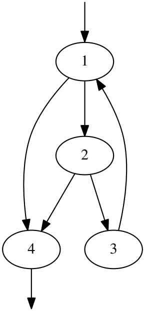

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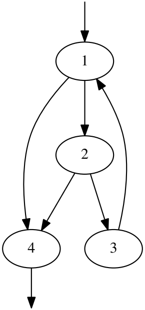

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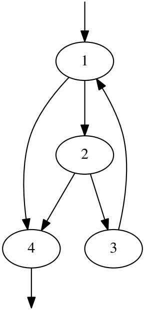

n	*count*(*n*)	*path*(*n*)
20	1	14
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24	0	16
25	1	17
26	1	18
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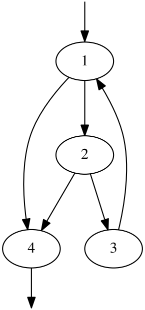

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Appears to grow linearly... is it $\frac{2}{3}n$?

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boolean passCheck2(){
 matched = true;
 while(i<n){
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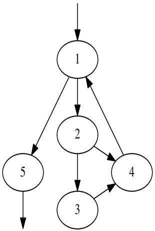

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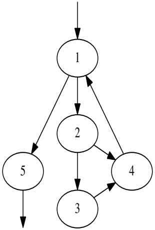

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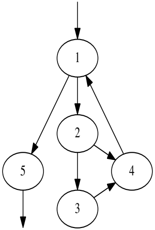

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21	16	85
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24	27	155
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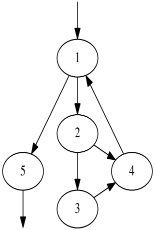

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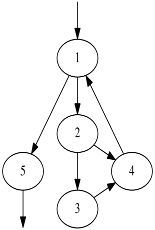

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Could be polynomial or exponential.

Computing Path Complexity

The path complexity problem:

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- How to compute *path*(n) **automatically**?

Computing Path Complexity

The path complexity problem:

- How to compute *path*(n) **automatically**?
- What is the **asymptotic behavior** of *path*(n)?

Matrix Exponentiation

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- For a particular n, we can compute *path*(n) using the $p \times p$ adjacency matrix, A, of the CFG, augmented with an additional 1 entry in the final column and final row.

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- For a particular n, we can compute $path(n)$ using the $p \times p$ adjacency matrix, A, of the CFG, augmented with an additional 1 entry in the final column and final row.
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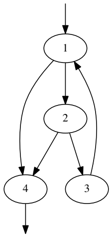

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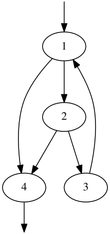

$$A^1 = \begin{bmatrix} 0 & 1 & 0 & \boxed{1} \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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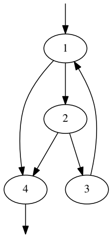

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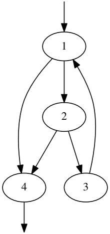

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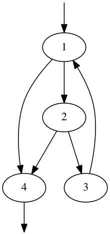

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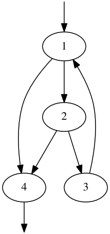

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$path(1) = 1$ $\quad$ $path(2) = 2$ $\quad$ $path(3) = 2$ $\quad$ $path(4) = 3$

Matrix Exponentiation

- For a particular n, we can compute $path(n)$ using the $p \times p$ adjacency matrix, A, of the CFG, augmented with an additional 1 entry in the final column and final row.
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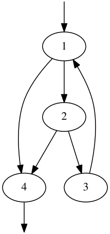

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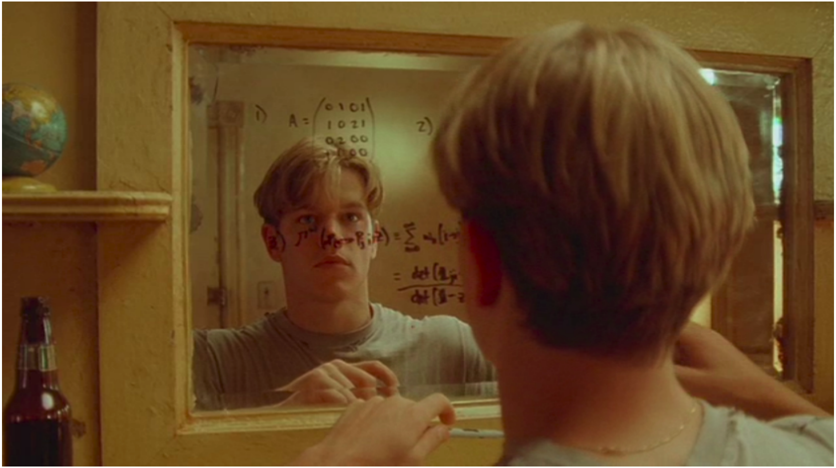
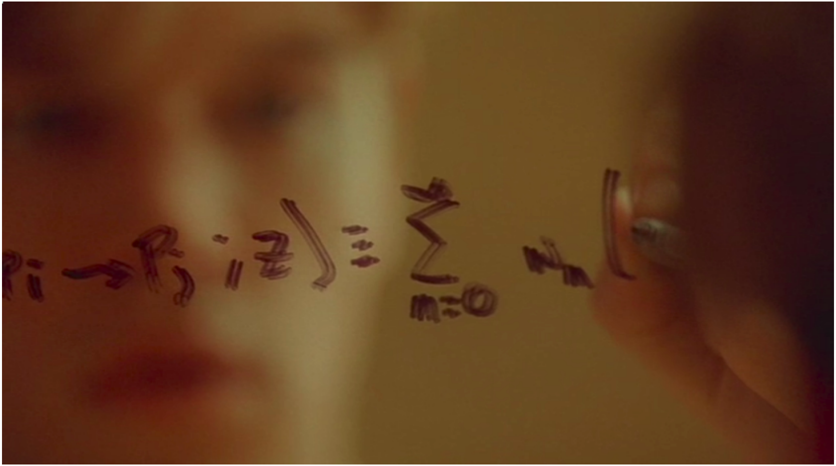
Drawback: repeated evaluations become expensive.

Matrix exponentiation works. Is there a better way?



Generating Functions



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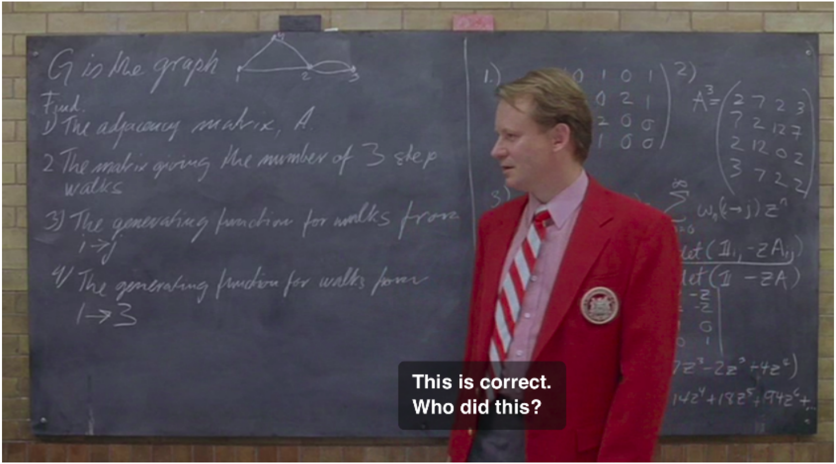
$$path(\boxed{6}) = \boxed{4}$$

Good job, Will Hunting!



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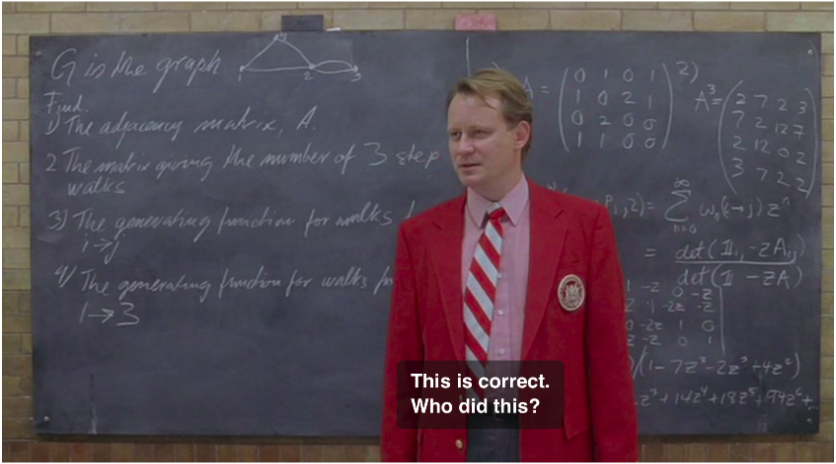

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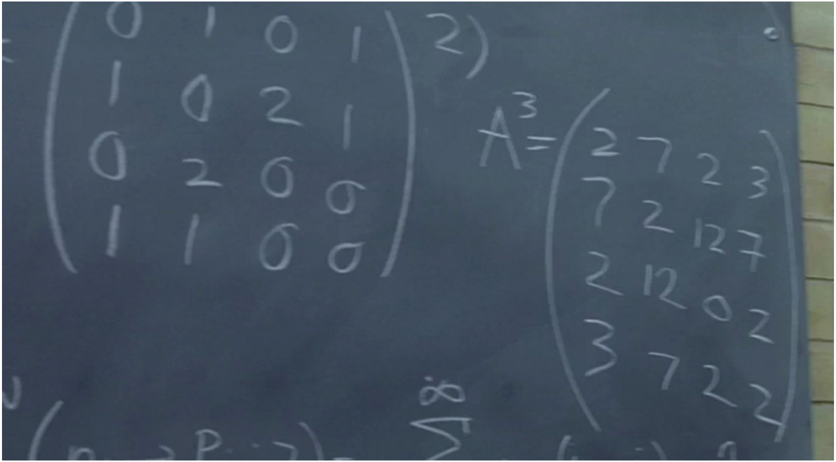

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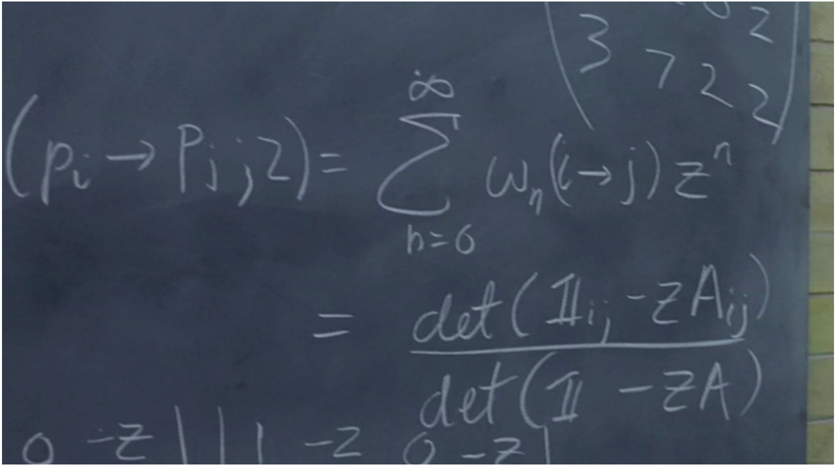

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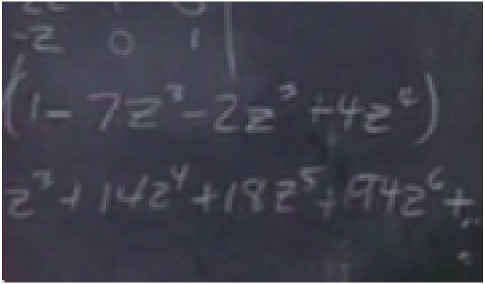

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- Solve for coefficients $c_1, \ldots, c_r$ using $g(z), g'(z), \ldots, g^{(r)}(z)$

$$path(n) = \frac{1}{3} + \frac{2}{3}n + \left(\frac{-3+\sqrt{3}}{18}\right)\left(\frac{-1+\sqrt{3}i}{2}\right)^n + \left(\frac{-3-\sqrt{3}}{18}\right)\left(\frac{-1-\sqrt{3}i}{2}\right)^n$$

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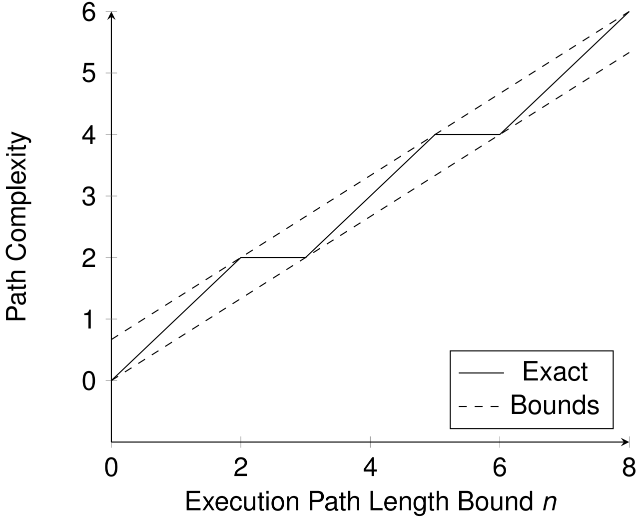
Now, it looks much simpler:

$$\frac{2n}{3} \leq path(n) \leq \frac{2n}{3} + \frac{2}{3}$$

Tight bounds for *path*(*n*)



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- We extract the highest order term using standard asymptotic analysis from calculus

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  - Function `passCheck2()`

$$path(n) = \Theta\left(1.221^n\right)$$

Complexity Classes

Classify path complexities as **constant**, polynomial, or exponential.

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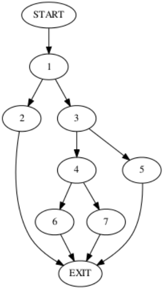

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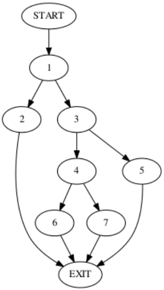

- Path Complexity: 4

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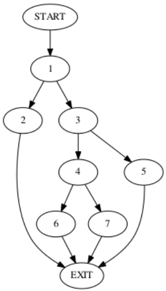

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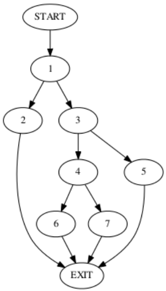

- Path Complexity: 4
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Classify path complexities as constant, **polynomial**, or exponential.

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public Matcher reset() {
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  for(int i=0; i<locals.length; i++)
    locals[i] = -1;
  lastAppendPosition = 0;
  from = 0;
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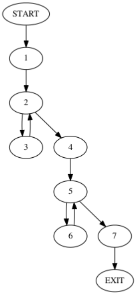

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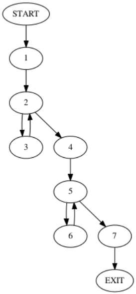

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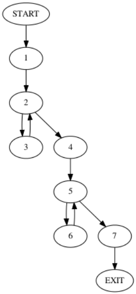

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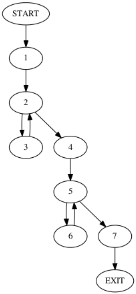

- Path Complexity: $0.12n^2 + 1.25n + 3$
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Complexity Classes

Classify path complexities as constant, polynomial, or **exponential**.

Examples from Java SDK 7.

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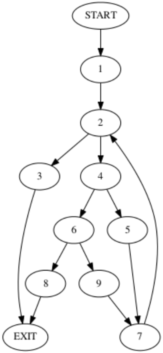

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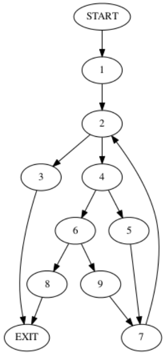

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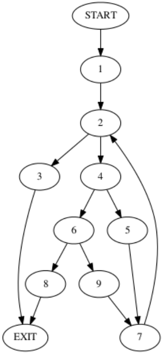

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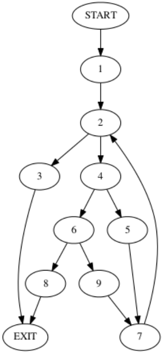

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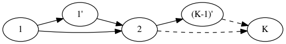
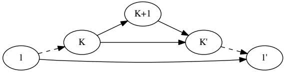
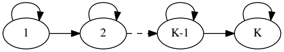
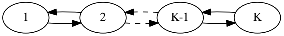
- Limitation: Both cyclomatic and NPATH return constant numbers, regardless of loops.
- Comparison of cyclomatic, NPATH, and path complexities.

Method	Cyclomatic Complexity	NPATH Complexity	Path Complexity	Asymptotic Complexity
rangeCheck()	4	4	4	$\Theta(1)$
reset()	3	4	$0.12n^2 + 1.25n + 3$	$\Theta(n^2)$
binarySearch0()	4	4	$(6.86)1.17^n + (0.22)1.1^n$ $+(0.13)(0.84)^n + 2$	$\Theta(1.17^n)$

Complexity Comparison

Pattern	Control Flow Graph	Cyclomatic Complexity	NPATH Complexity	Asymptotic Complexity
K If-Else in sequence		$K+1$	2^K	2^K
K If-Else nested		$K+1$	$K+1$	$K+1$
K Loops in sequence		$K+1$	2^K	$\Theta(n^K)$
K Loops nested		$K+1$	$K+1$	$\Theta(b^n)$

Experiments

- Tested our analysis on Java 7 SDK (132K methods, $\approx$ 2.5 hr.) and Apache Commons (44K methods, $\approx$ 1 hr.) libraries.
- Separated methods into complexity classes:
  - $C = 1$ Unique path
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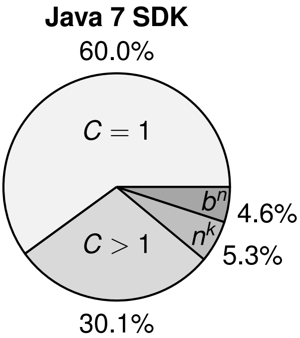

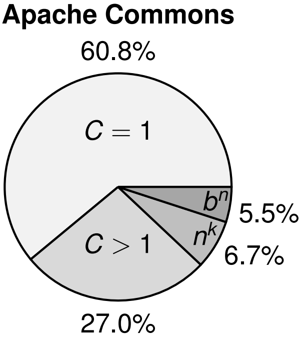

Replication Package

Our tool is called PAth Complexity Analyzer (PAC).

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- Web version.
  1. Upload Java `.class` or `.jar` file.
  2. Output a table of cyclomatic, NPATH, and (asymptotic) path complexities for all methods.

Future Work

- Experimentally validate that path complexity is a good measure of the difficulty of acheiving path coverage.
- Extend analysis to inter-procedural calls using the theory of generating functions for generative grammars.
- Path complexity may count infeasible paths–provides only an upper bound. Refine path complexity to consider simple path conditions.
- Apply path complexity results to side-channel analysis for timing attacks.

Thank you.