R–LINE: An Online Algorithm for the 2-Server Problem on the Line with Improved Competitive Ratio

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Thesis Defense

Abstract

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- A randomized online algorithm for the 2-server problem on the line.

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- A randomized online algorithm for the 2-server problem on the line.
- Competitiveness ≤ 1.901 against the oblivious adversary.
- Improves the previously best known competitiveness of $\frac{155}{78} \approx 1.987$.

Outline

1. Offline vs. online algorithms.
2. Competitive analysis and game theory.
3. The *k*-server problem.
4. Our 2-server algorithm, R–LINE (Randomized Line).
5. Future work.

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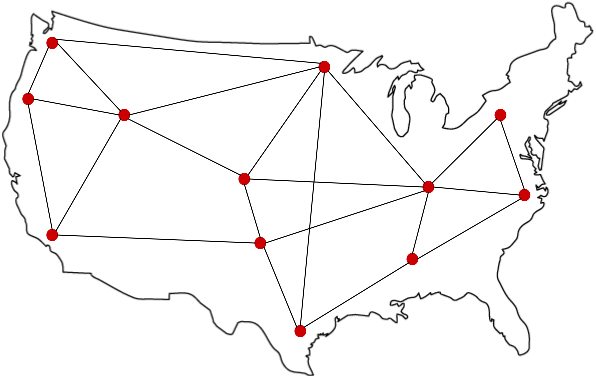

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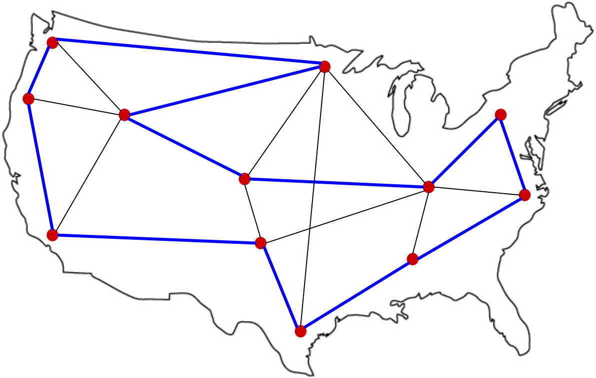

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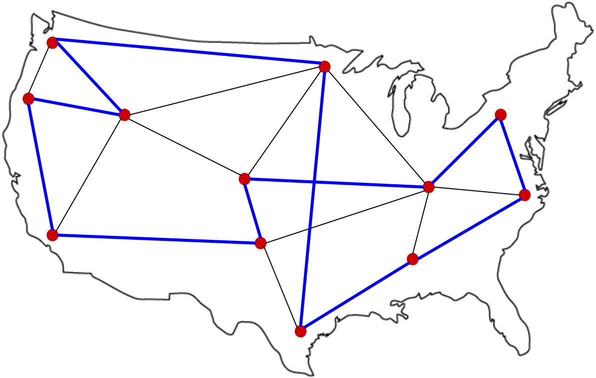

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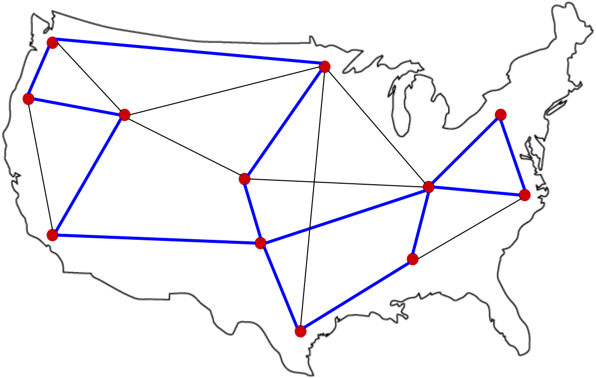

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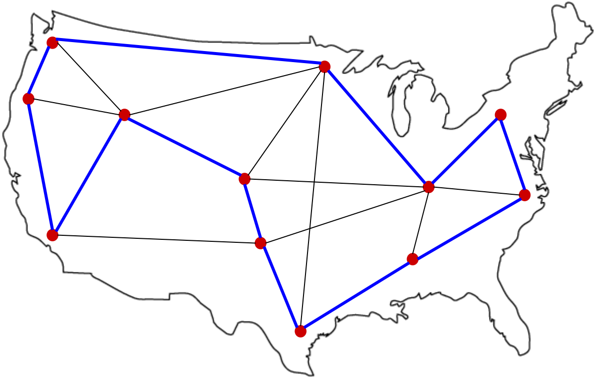

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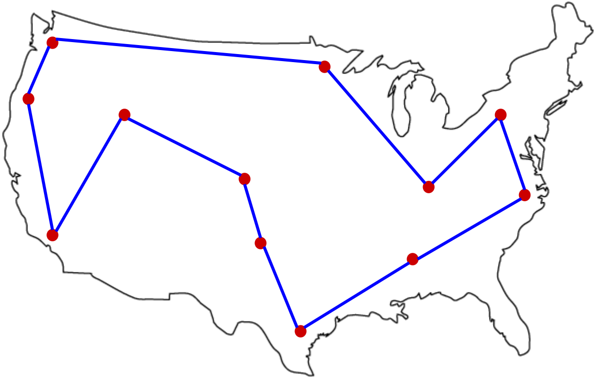

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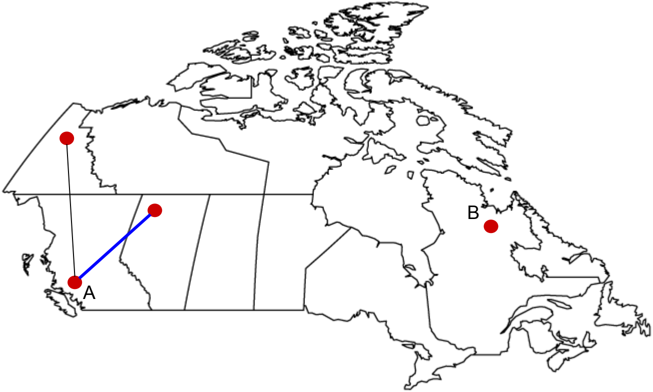

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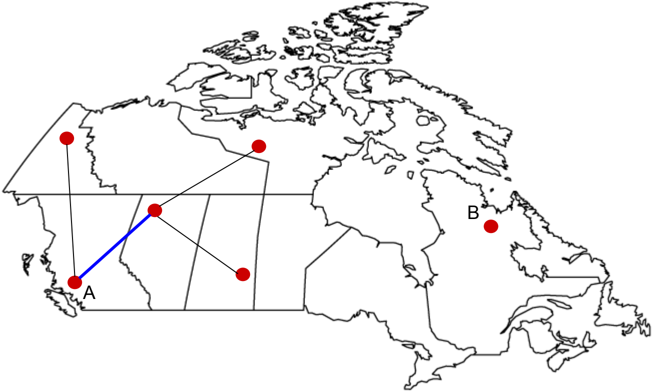

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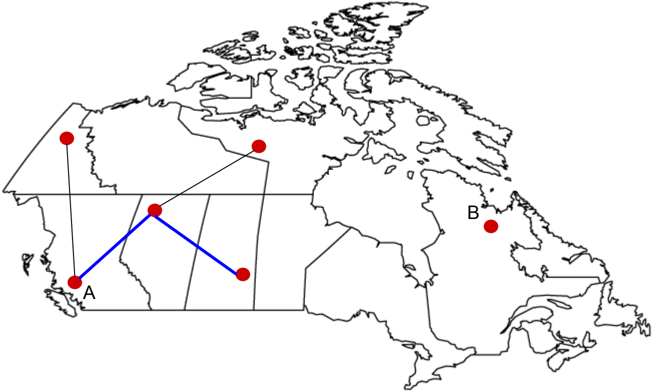

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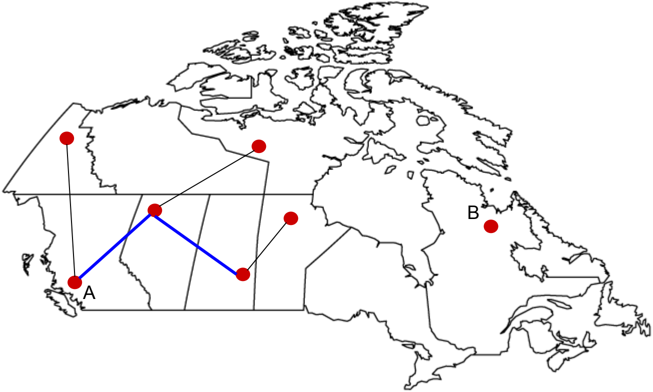

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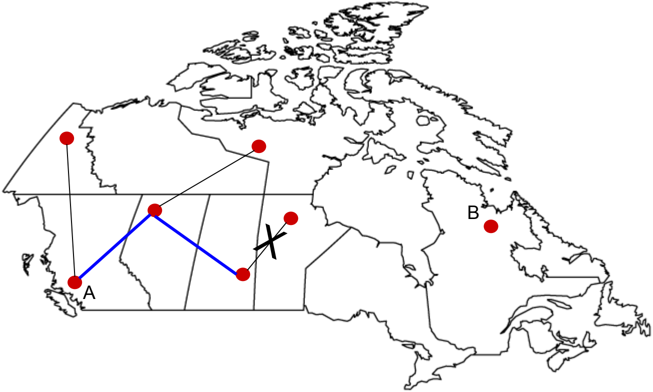

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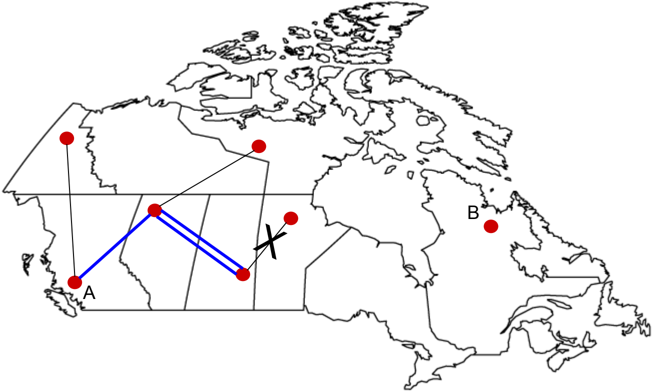

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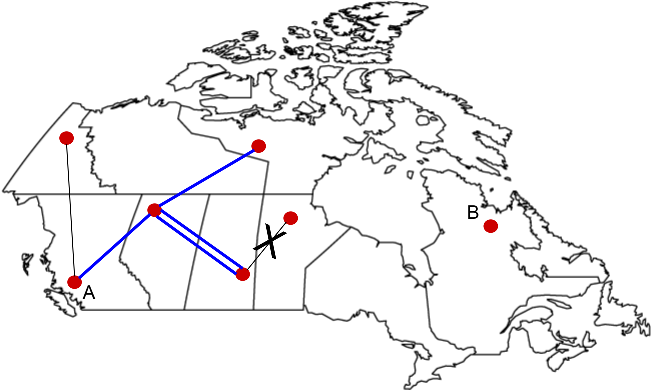

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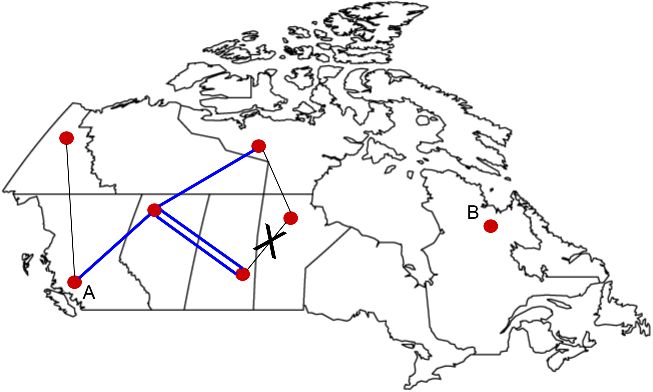

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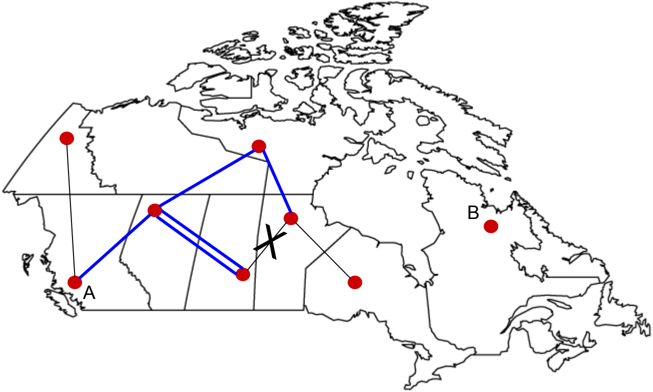

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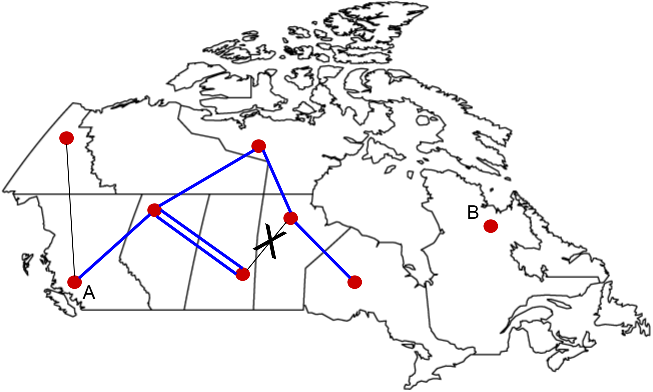

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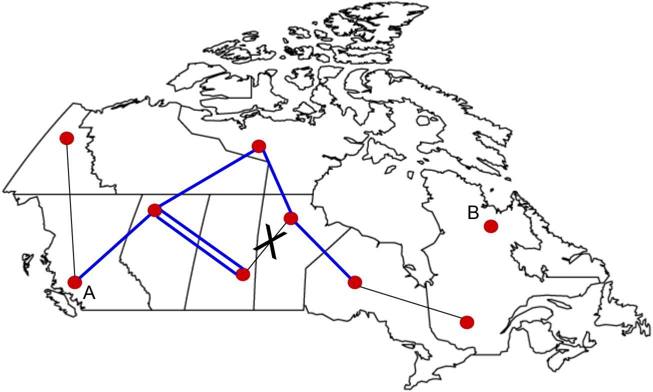

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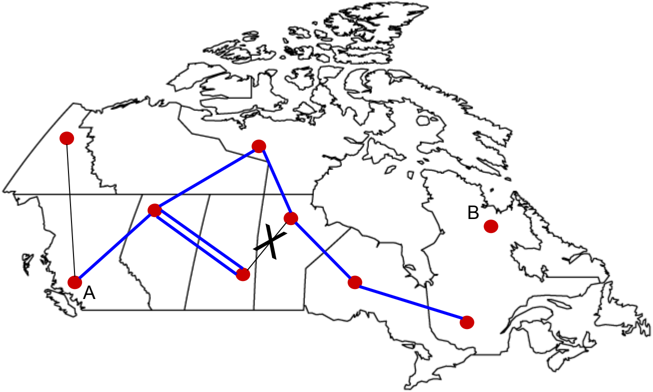

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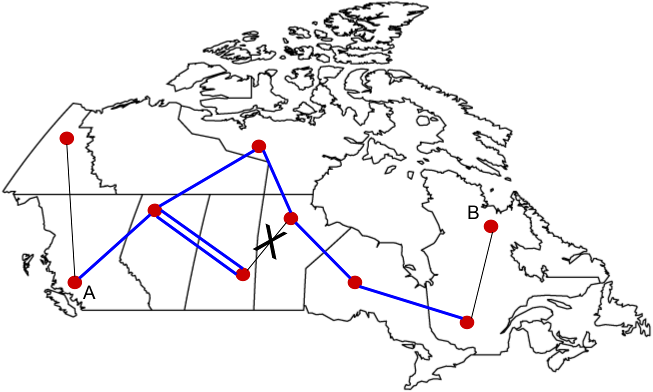

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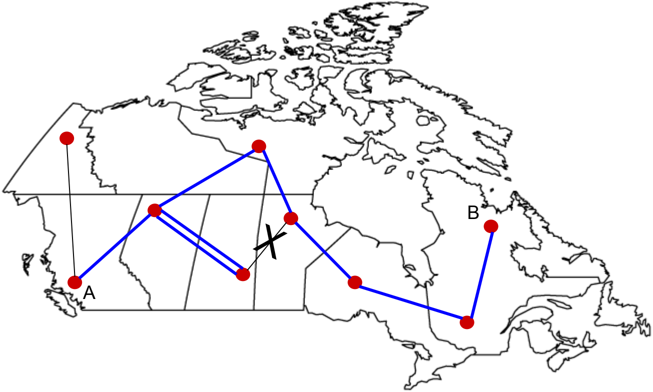

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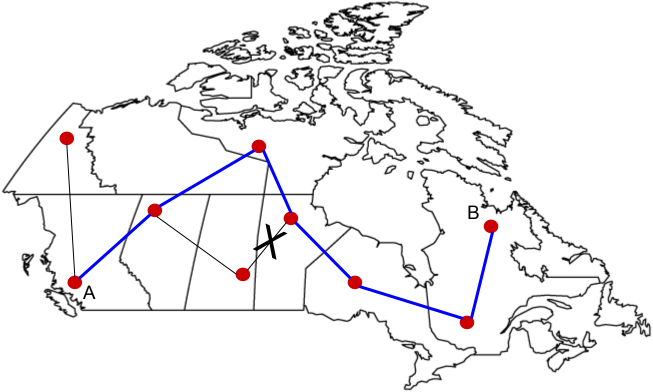

Online Algorithms

Many important real-world problems are online.

Examples

1. Investment decisions, as in algorithmic stock trading.
2. Job scheduling, as in multi-core computing.
3. Memory cache page management.
4. and more....

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2. Receive an input I_i, and produces an output O_i.
3. Each input has an associated $\text{cost}(I_i)$.
4. We wish to minimize the total $\text{cost}(I) = \sum \text{cost}(I_i)$.

Competitive Analysis

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- For any request sequence I the competitive ratio C satisfies

$$cost_A(I) \leq C \cdot cost_{\mathcal{OPT}}(I) + K$$

- If A is randomized then

$$E[cost_A(I)] \leq C \cdot cost_{\mathcal{OPT}}(I) + K$$

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The k-server Problem

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- Goal: minimize total distance moved.

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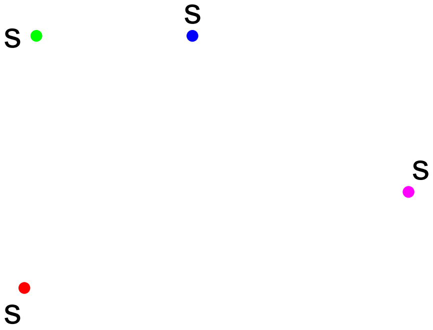

$cost = \ldots$

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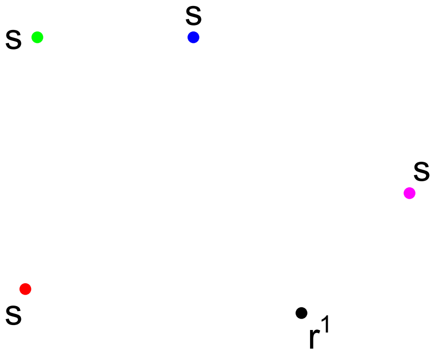

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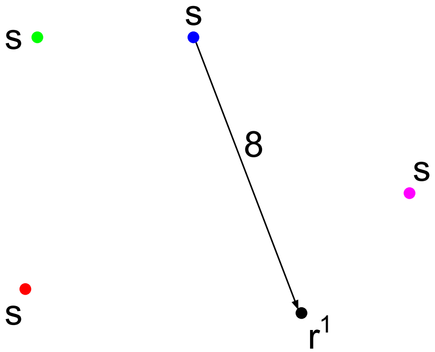

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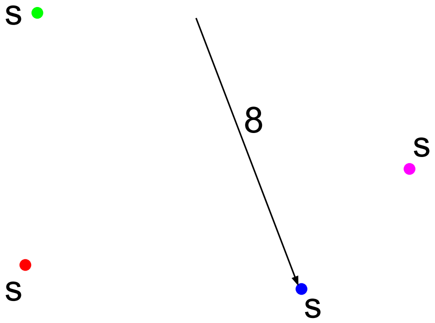

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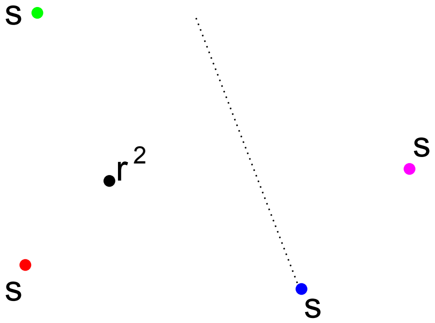

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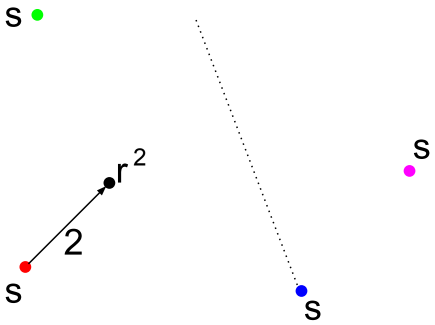

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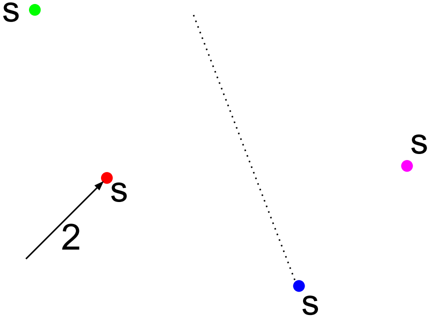

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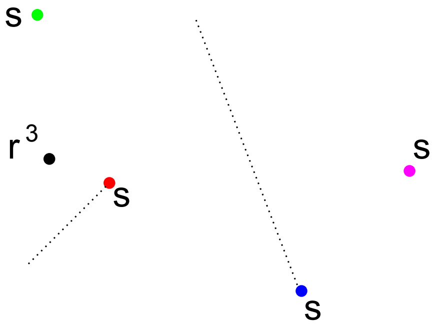

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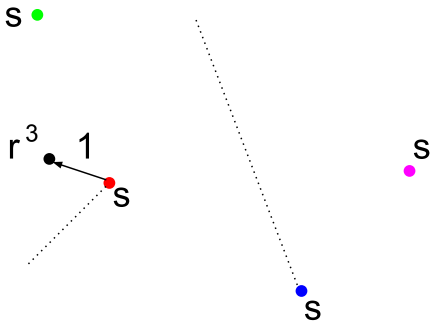

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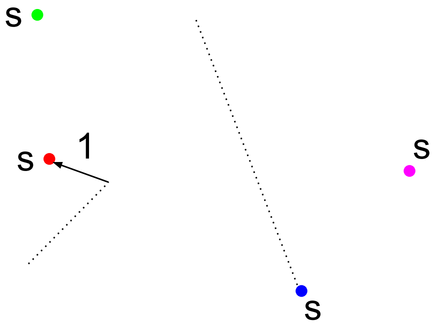

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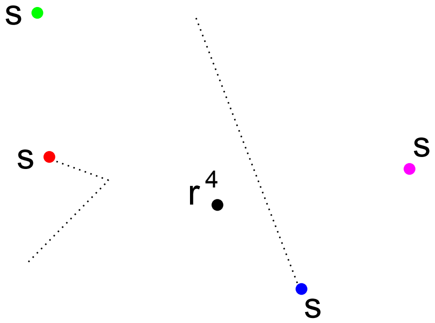

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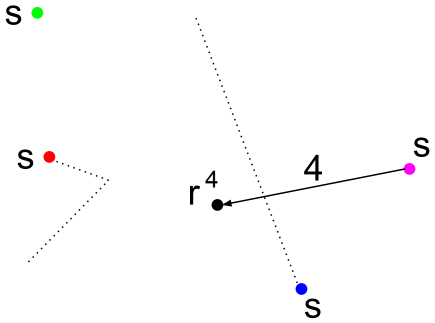

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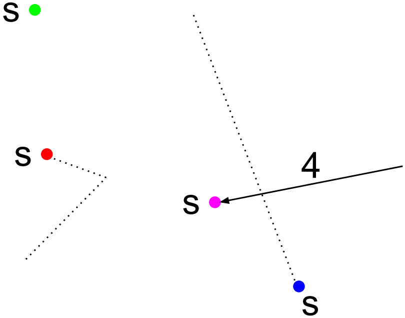

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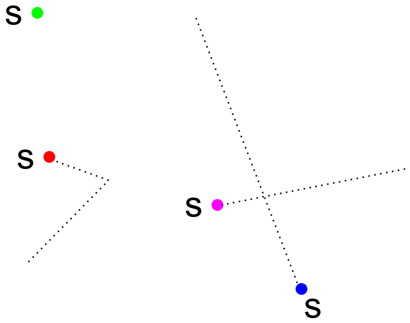

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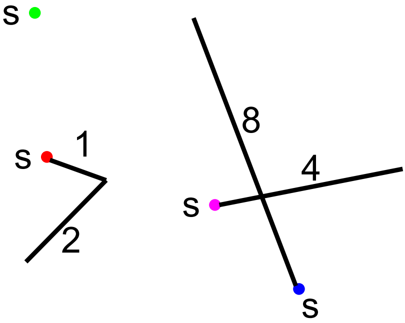

$cost = 8 + 2 + 1 + 4 = 15$

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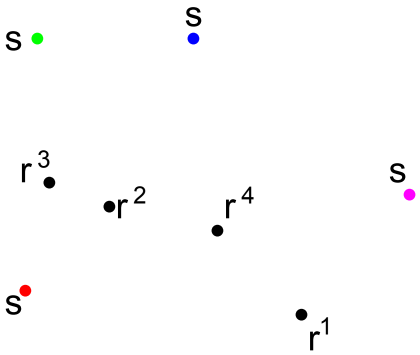

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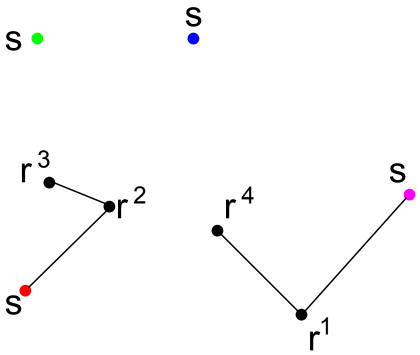

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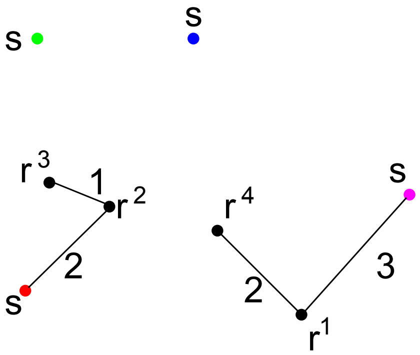

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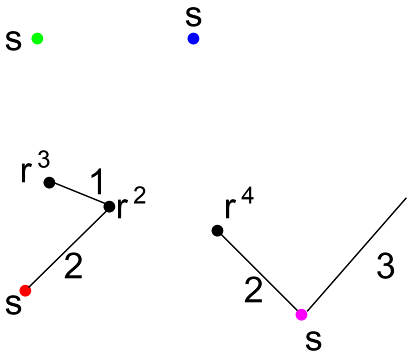

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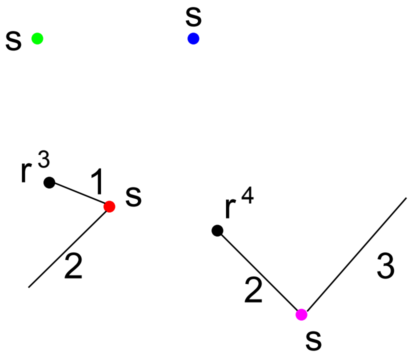

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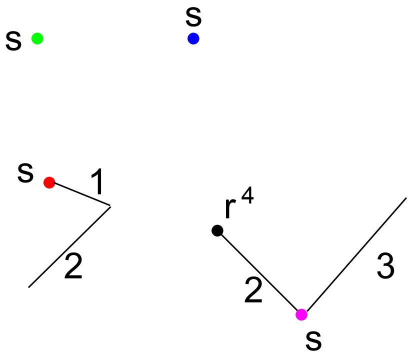

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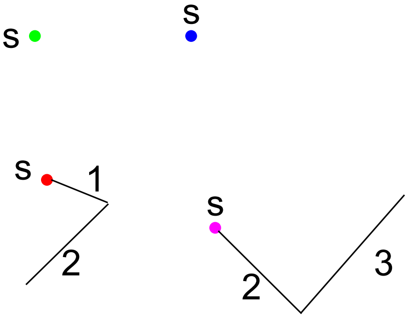

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If this holds for **all possible** input sequences, we could claim that A is 2-competitive.

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$$cost_A(R) \leq C \cdot cost_{OPT}(R) + K.$$

The Optimal Adversary Algorithm

How to model the optimal algorithm?

- We think of the optimal algorithm as a malevolent adversary.
- Adversary generates the input sequence $I_1, I_2, \ldots, I_n$
- Adversary must also use its servers to satisfy requests.
- Adversary tries to maximize C by simultaneously making its cost low and our cost high.

Game Theory

We can now think of the server problem as a game between our algorithm and the adversary algorithm. Consider the payoff matrix for a two-person zero-sum game G.

	Adversary Strategy 1	Adversary Strategy 2
Server Strategy 1	a_{11}	a_{12}
Server Strategy 2	a_{21}	a_{22}

Game Theory

We can now think of the server problem as a game between our algorithm and the adversary algorithm. Consider the payoff matrix for a two-person zero-sum game G.

	Adversary Strategy 1	Adversary Strategy 2
Server Strategy 1	a_{11}	a_{12}
Server Strategy 2	a_{21}	a_{22}

$$v(G) = \frac{\det A}{a_{11} - a_{12} - a_{21} + a_{22}}$$

Game Theory

	Adversary Strategy 1	Adversary Strategy 2
Server Strategy 1	a_{11}	a_{12}
Server Strategy 2	a_{21}	a_{22}

Optimum **row player** strategy:

Play row 1 with $p_1 = \frac{a_{22} - a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}}$

Play row 2 with $p_2 = \frac{a_{11} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}}$

Optimum **column player** strategy:

Play column 1 with $p_1 = \frac{a_{22} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}}$

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Generalization of the k-server problem

The (m, n)-server Problem

Generalization of the k-server problem

The (m, n)-server Problem

- Given m mobile servers in a metric space M.

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The (m, n)-server Problem

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- Goal: minimize total distance moved.

Background

$(4, 2)$-server Problem

Background

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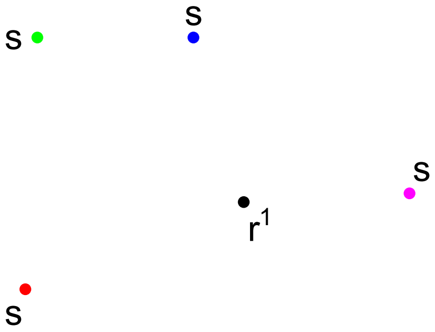
Background

$(4,2)$-server Problem



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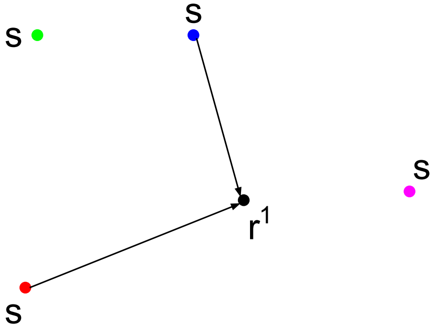

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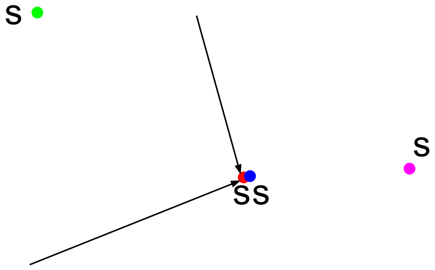

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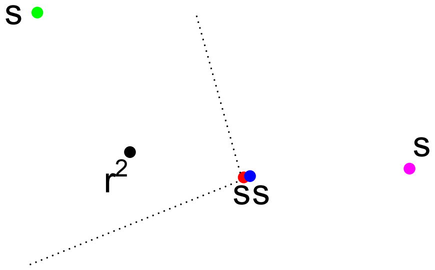

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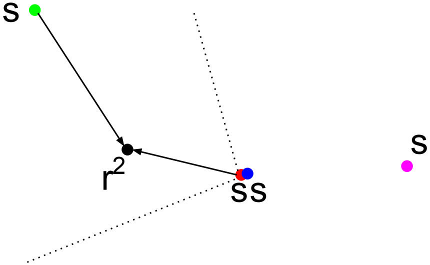

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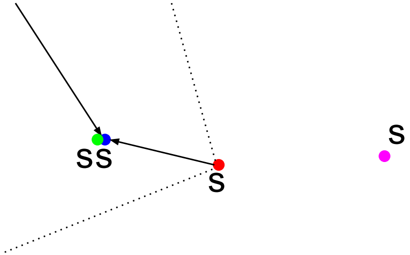

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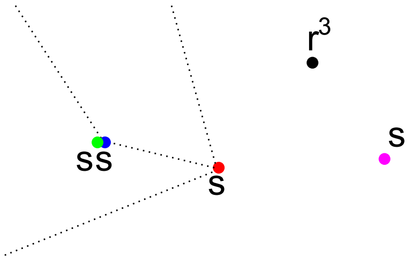

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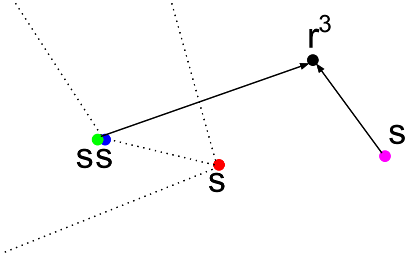

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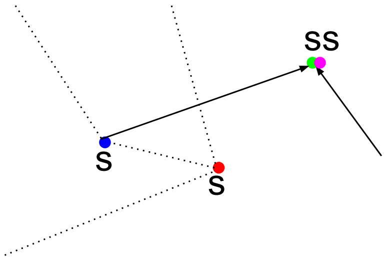

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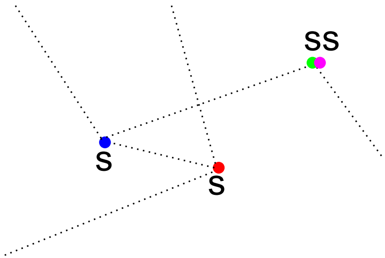

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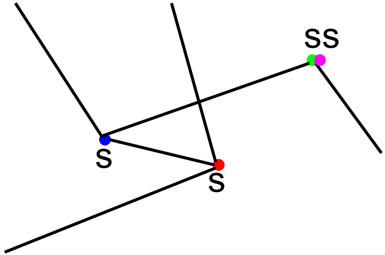

Useful Results

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Theorem 1

$$C\text{-competitive } (2n, n)\text{-server Algorithm}$$
$$\Downarrow$$
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$$\begin{gathered}C\text{-competitive } (2n, n)\text{-server Algorithm} \\ \Downarrow \\ C\text{-competitive 2-server Algorithm}\end{gathered}$$

Theorem 2

Optimal offline strategy for the $(2n, n)$ server problem keeps the servers in two blocks of size n.

R–LINE

We give an online algorithm for the 2-server problem where the metric space is the real line.

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Outline

- Define a randomized online algorithm for the $(2n, n)$-server problem.

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- For R–LINE, $C \leq 1.901$

R–LINE Details

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- $s_0 = a_0 = -\infty$ and $s_{2n+1} = a_3 = \infty$

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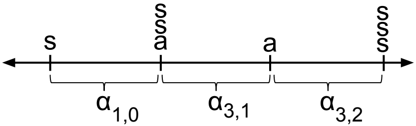

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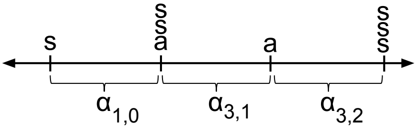

$$\alpha_{3,0} = 0, \alpha_{4,1} = 0, \ldots$$

R–LINE Details

Isolation Index Coefficients

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$$\eta_{i,j} = \eta_{2n-i,2-j}$$

- We also have $\eta_{0,0} = \eta_{2n,n} = 0$.

R–LINE Details

Definition of the Potential, ϕ

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- For any configuration, the potential is defined as the sum of all isolation indices multiplied by their associated coefficients.
- Formally,

$$\phi = \sum \eta_{i,j} \cdot \alpha_{i,j}$$

R–LINE Details

Configurations

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Notation

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- We refer to s_i as the i^{th} server and also its location.

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- Servers do not pass each other.

R–LINE Details

S-Configuration (Satisfying)

R–LINE Details

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R–LINE Details

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- There are n servers at the request point.

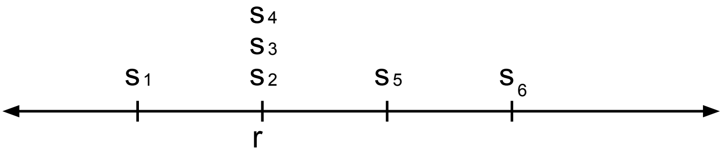
$(6,3)$ Example

R–LINE Details

S-Configuration (Satisfying)

- There are *n* servers at the request point.

$(6, 3)$ Example



R–LINE Details

D-Configuration (Deterministic)

R–LINE Details

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- 1. More than n algorithm servers either strictly to the left or strictly to the right of r; $r > s_{n+1}$ or $r < s_n$.
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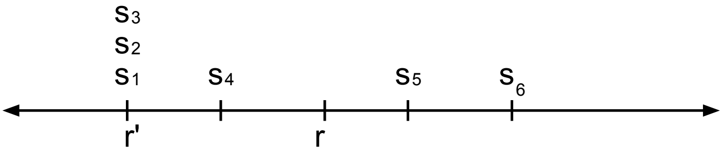
$(6, 3)$ Example



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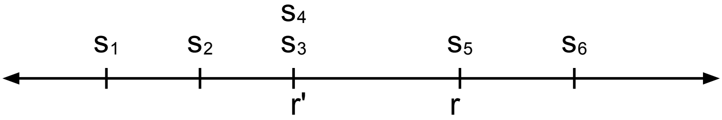

R–LINE Details

D-Configuration Moves

1. Must be m servers to the left of r, for some $m > n$.

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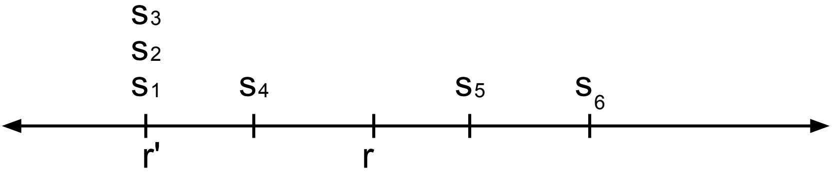

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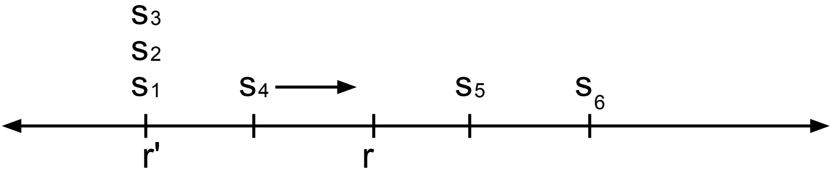

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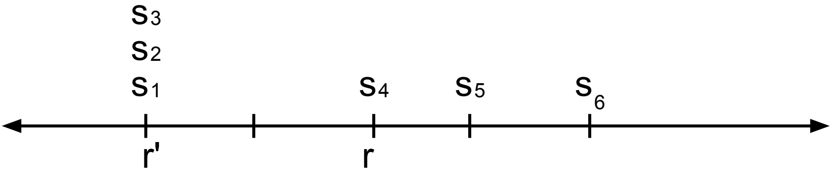

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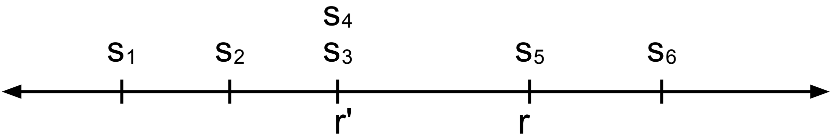

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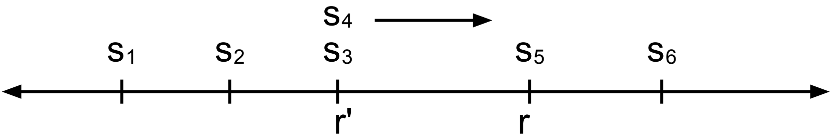

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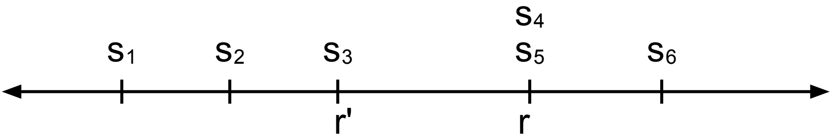

R–LINE Details

R-Configurations (Randomized)

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The adversary's hidden server.

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- The adversary has two servers.

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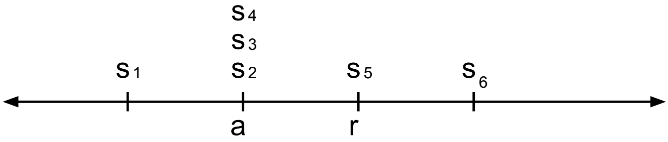

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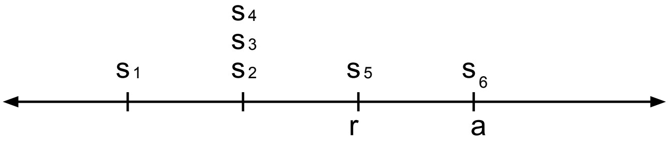

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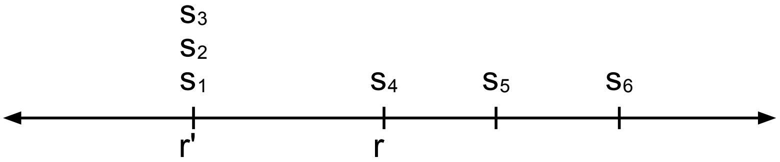

R–LINE Details

R-Configuration (Randomized) Moves

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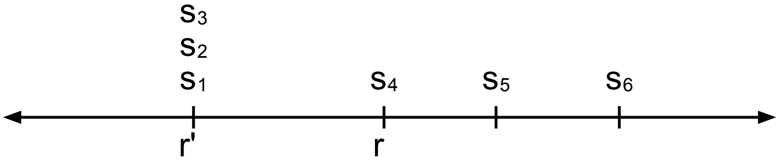

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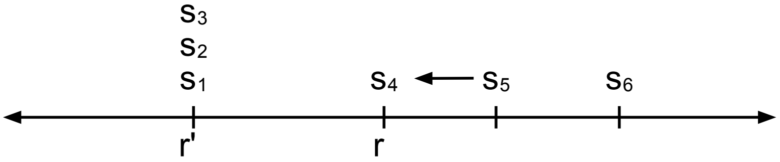

R–LINE Details

R-Configuration (Randomized) Moves

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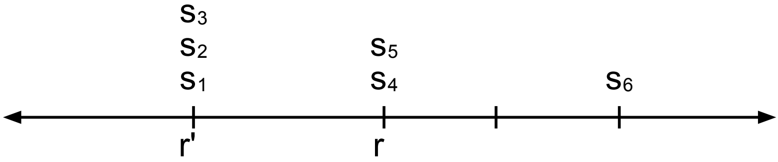

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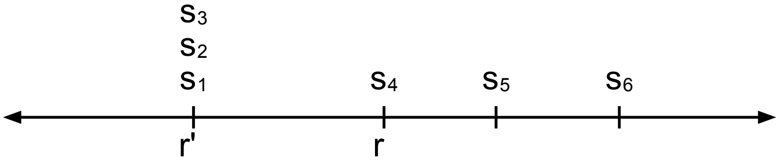

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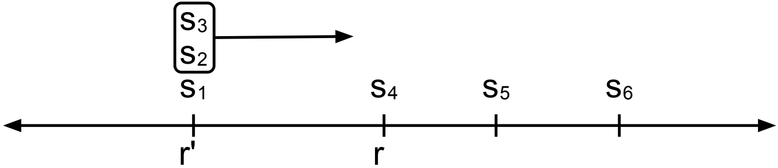

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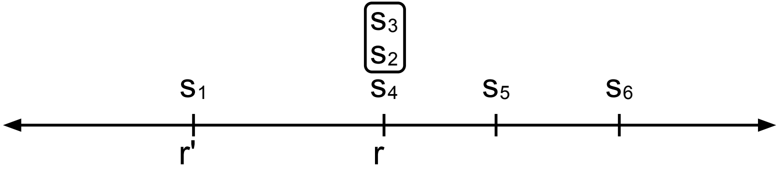

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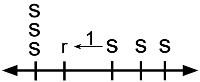
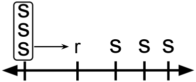
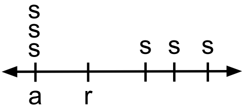
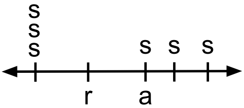
		
	$(\Delta\phi + cost)_{11}$	$(\Delta\phi + cost)_{12}$
	$(\Delta\phi + cost)_{21}$	$(\Delta\phi + cost)_{22}$

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$$
\begin{aligned}
(\Delta\phi + cost)_{11} &= (\eta_{n+p+1,2} - \eta_{n+p,2} + 1) \cdot (s_{n+p+1} - r) \\
(\Delta\phi + cost)_{12} &= (\eta_{p,1} - \eta_{n,1} + n - p) \cdot (r - s_n) \\
(\Delta\phi + cost)_{21} &= (\eta_{n+p+1,1} - \eta_{n+p,1} + 1) \cdot (s_{n+p+1} - r) \\
(\Delta\phi + cost)_{22} &= (\eta_{p,0} - \eta_{n,0} + n - p) \cdot (r - s_n)
\end{aligned}
$$

R–LINE Details

The Algorithm R–LINE

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For a given round of execution:

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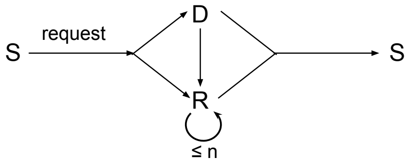
For a given round of execution:

1. Start in S-Config. Receive a request.
2. If D–Config, make deterministic moves.
   2.1 If result is S–Config, done.
   2.2 Otherwise result is R–Config.
3. If R–Config, make randomized moves until S–Config.



Proof of Competitiveness

Overview of Proof

1. Provide a system of inequalities, $\mathbb{S}$, involving the isolation index coefficients, $\eta_{i,j}$, and the competitiveness, C.
2. Show that if there exists an assignment of values to every $\eta_{i,j}$ that satisfies $\mathbb{S}$, then R–LINE is C-competitive.
3. Use numeric methods to find a solution to $\mathbb{S}$ that minimizes C.

Proof of Competitiveness

Sufficient Inequalities, $\mathbb{S}_n$

$$\begin{aligned}
\forall\, 0 \le i \le 2n:\ |\eta_{i,1} - \eta_{i,0}| &\le n \cdot C && (1)\\
\forall\, 1 \le i \le n \text{ and } \forall\, 1 \le j \le 2:\ \eta_{i,j} + 1 &\le \eta_{i-1,j} && (2)\\
\forall\, 1 \le i \le n \text{ and } \forall\, 1 \le j \le 2:\ \eta_{i-1,j-1} &\le \eta_{i,j-1} + 1 && (3)\\
\forall\, 1 \le i \le n:\ (\eta_{i-1,1} - \eta_{i,1} + 1)(\eta_{n-i,1} - \eta_{n,1} + i) &\le (\eta_{i-1,0} - \eta_{i,0} + 1)(\eta_{n-i,0} - \eta_{n,0} + i) && (4)
\end{aligned}$$

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Adding all of the inequalities over a request round:

$$E(cost_A(\sigma)) \leq C \cdot cost_{\mathcal{ADV}}(\sigma) + K$$

Solving the Inequalities

Given the system $\mathbb{S}_n$

- Find a solution to $\mathbb{S}_n$ that minimizes C.

Solving the Inequalities

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- Find a solution to $\mathbb{S}_n$ that minimizes C.
- Transform $\mathbb{S}_n$ into a new system $\mathbb{S}'_n$ to simplify the solution.

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Introduce variables ϵ_i and δ_i for all $0 \leq i < n$. Then $\mathbb{S}'_n$ consists of the following constraints:

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$$
\begin{array}{rcll}
(2i + \epsilon_{n-i})(2 - \epsilon_i + \epsilon_{i-1}) & = & 4i & \forall\ 0 < i < n \\
(2n + \epsilon_0)(2 + \epsilon_{n-1}) & \geq & 4n & \\
\delta & = & -\epsilon_0/2 & \\
C & = & (2n - \delta)/n & \\
\delta_i & = & \epsilon_i + 2\delta & \forall\ 0 \leq i < n \\
\eta_{i,0} & = & 3i - \delta_i & \forall\ 0 \leq i < n \\
\eta_{i,0} & = & 2n + i - 2\delta & \forall\ n \leq i \leq 2n \\
\eta_{i,1} & = & 2n - i - \delta & \forall\ 0 \leq i < n \\
\eta_{i,1} & = & i - \delta & \forall\ n \leq i \leq 2n \\
\eta_{i,2} & = & \eta_{2n-i,0} & \forall\ 0 \leq i \leq 2n
\end{array}
$$

Solving the Inequalities

Converting to a Differential Equation

- As $n \to \infty$, $\mathbb{S}'_n \to D$
- D is given by:

$$(x + 1 + f(-x)) \cdot (1 - f'(x)) = x + 1$$

$$-1 \leq x \leq 1$$

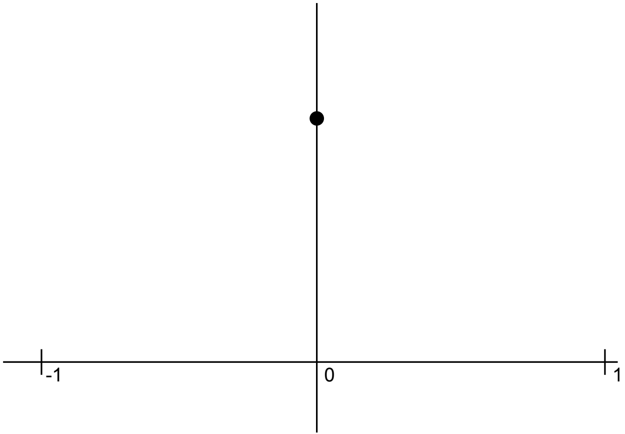
Solving the Inequalities

Euler Method for Reflective Differential Equation

1. Choose a step size, $h = \frac{2}{n}$.
2. Choose an initial value y_0 at t_0
3. Compute updates:

$$y_{n+1} = y_n + h \cdot f(t_{-n}, y_{-n}),$$
$$y_{-n-1} = y_{-n} + h \cdot f(t_n, y_n),$$

Solving the Inequalities



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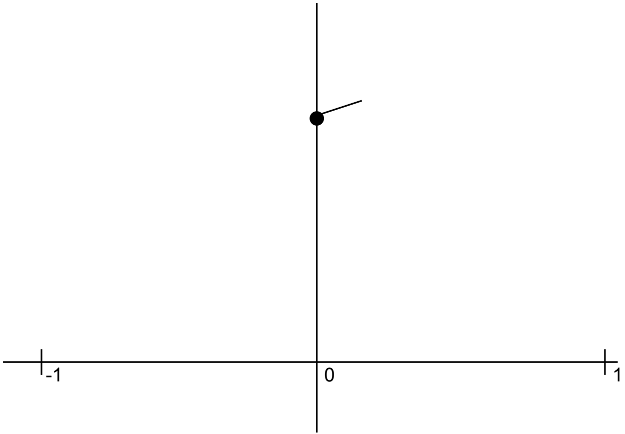

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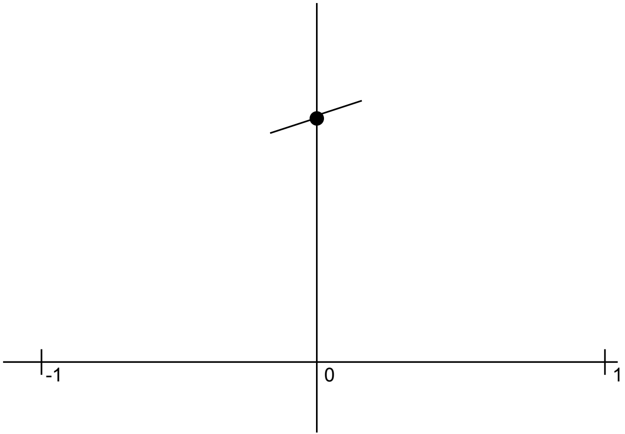

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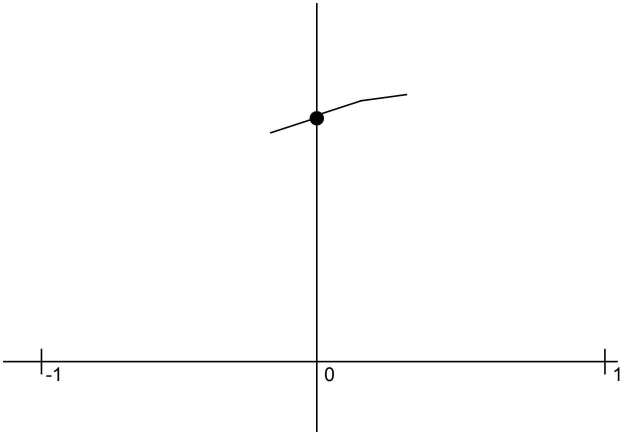

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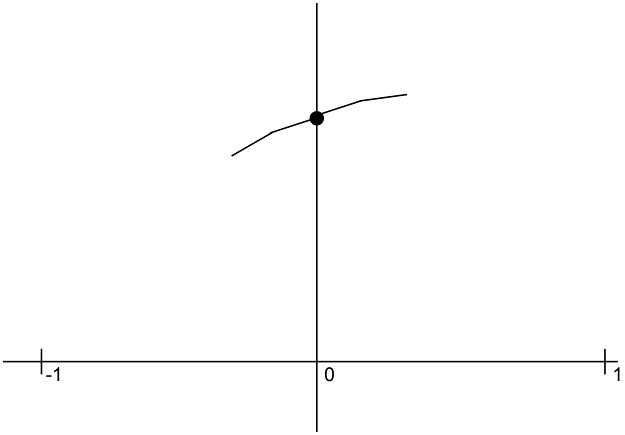

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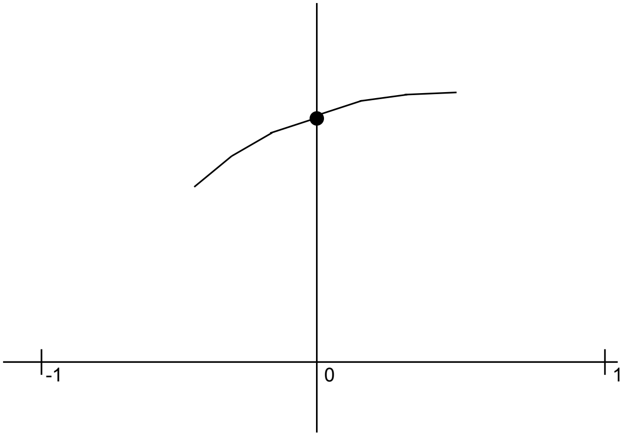

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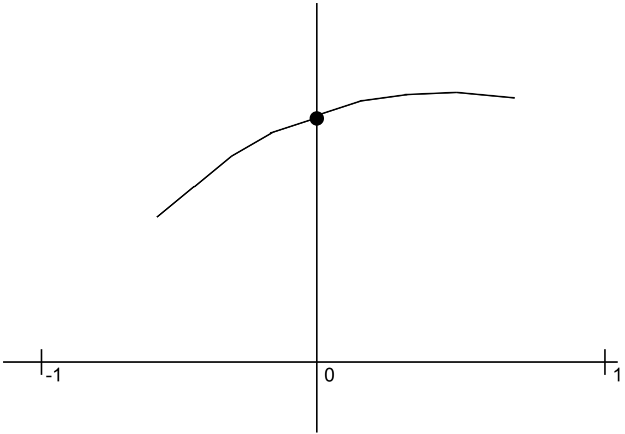

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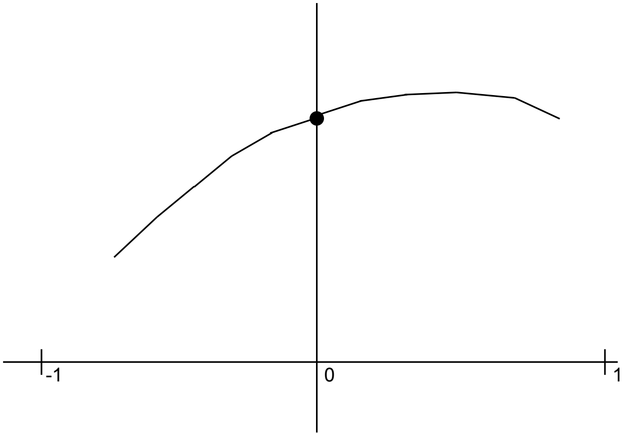

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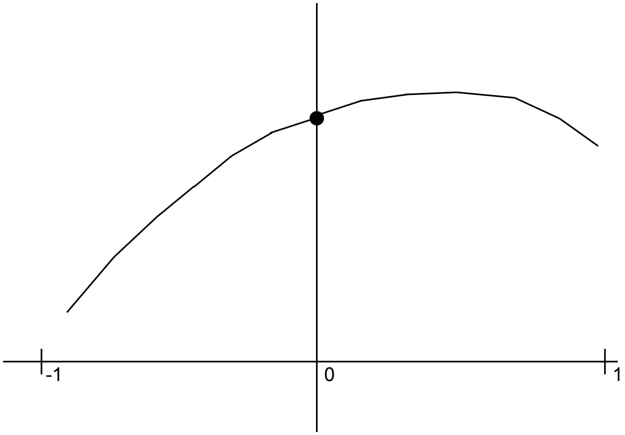

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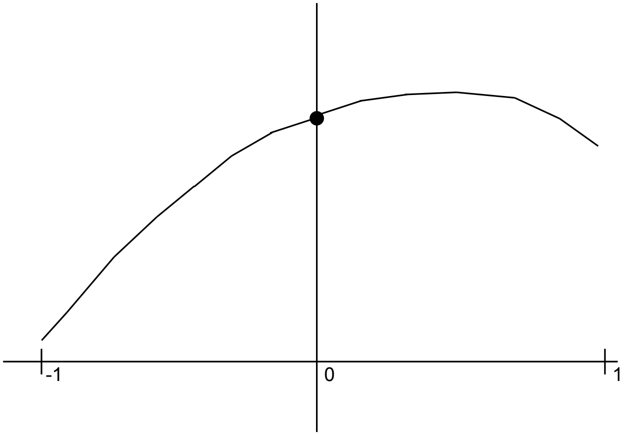


Solving the Inequalities

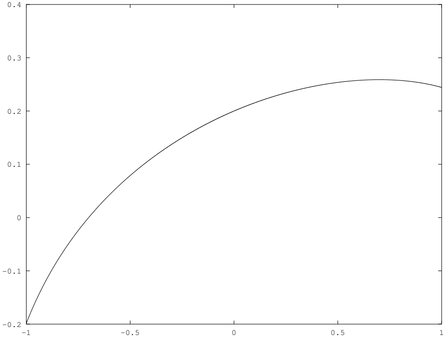
Algorithm to Find Minimum C

1. Choose initial value $f(0)$.
2. Approximate f on interval $[-1, 1]$.
3. Use substitutions to find $\eta_{i,j}$.
4. Verify that $\eta_{i,j}$ satisfy $\mathbb{S}_n$.
5. Compute corresponding value of C.
6. Binary search on $f(0)$ to find minimum C.

Solving the Inequalities

Solution for large n

- We find that $C \approx 1.9007452$ for $n = 10000$.
- The corresponding approximation of $f(x)$:



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Analytic Solution to the Differential Equation

- Would give exact minimum value of C for R–LINE.

Acknowledgements

Thank you!

- Committee Members: Lawrence Larmore, Wolfgang Bein, Matt Pedersen, Ebrahim Salehi.
- UNLV Graduate and Professional Student Association.
- Attendees.

R–LINE

The End.

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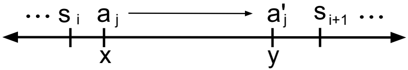

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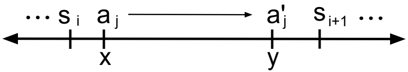

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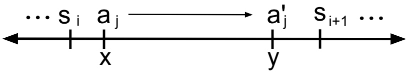

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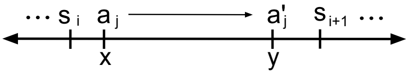

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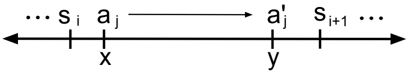

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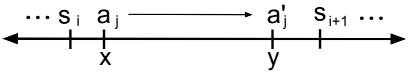

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- By definition of the potential, $\Delta\phi = (\eta_{i,j} - \eta_{i,j-1})(y - x) \leq n \cdot C \cdot (y - x) \leq C \cdot cost_{\mathcal{A}dv}$.

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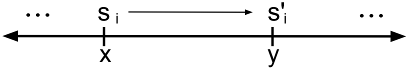

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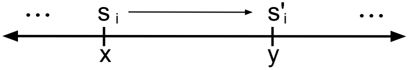

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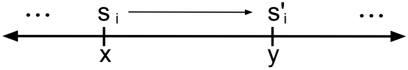

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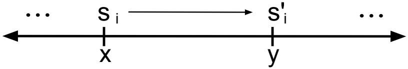

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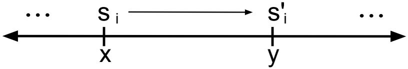

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- The move causes $\alpha_{i,j}$ to decrease by $y - x$ and $\alpha_{i-1,j}$ to increase by the same amount.
- By inequality (2), $\eta_{i,j} + 1 \leq \eta_{i-1,j}$, and the definition of the potential: $\Delta\phi + cost_{R--LINE} = (y - x)(\eta_{i,j} - \eta_{i-1,j} + 1) \leq 0$.

Proof of Competitiveness

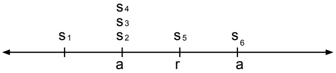
Proof of Property 3

If $\mathbb{S}$ holds, then Property (3) holds: We may assume the adversary's hidden server is at one of at most two possible locations.

Proof of Competitiveness

Proof of Property 3

If $\mathbb{S}$ holds, then Property (3) holds: We may assume the adversary's hidden server is at one of at most two possible locations.



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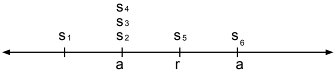

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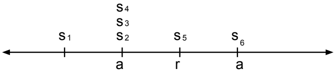

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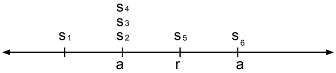

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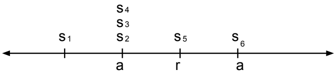

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		Move s_{n+p+1}	Move $s_{p+1} \dots s_n$
I	$a \le s_n$	$\left(\eta_{n+p+1,2} - \eta_{n+p,2} + 1\right)\left(s_{n+p+1} - r\right)$	$(\eta_{p,1} - \eta_{n,1} + n - p)(r - s_n)$
II	$s_n \le a \le r$	$\left(\eta_{n+p+1,2} - \eta_{n+p,2} + 1\right)\left(s_{n+p+1} - r\right)$	$(\eta_{p,1} - \eta_{n,1} + n - p)(r - a)$ $+$ $(\eta_{p,0} - \eta_{n,0} + n - p)(a - s_n)$
III	$r \le a \le s_{n+p+1}$	$\left(\eta_{n+p+1,2} - \eta_{n+p,2} + 1\right)\left(s_{n+p+1} - a\right)$ $+$ $(\eta_{n+p+1,1} - \eta_{n+p,1} + 1)(a - r)$	$(\eta_{p,0} - \eta_{n,0} + n - p)(r - s_n)$
IV	$a \ge s_{n+p+1}$	$\left(\eta_{n+p+1,1} - \eta_{n+p,1} + 1\right)\left(s_{n+p+1} - r\right)$	$(\eta_{p,0} - \eta_{n,0} + n - p)(r - s_n)$

By inequalities (2) and (3), the rows $a = s_n$ and $a = s_{n+p+1}$ dominate all other rows.

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$a = s_{n+p+1}$	$(\eta_{n+p+1,1} - \eta_{n+p,1} + 1)(s_{n+p+1} - r)$	$(\eta_{p,0} - \eta_{n,0} + n - p)(r - s_n)$

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- The value of our game is

$$\frac{\det(G)}{(\eta_{n+p+1,2} + \eta_{n+p+1} - \eta_{n+p,2} - \eta_{n+p+1,1}) \cdot (s_{n+p+1} - r) + (\eta_{p,0} + \eta_{n,1} - \eta_{n,0} - \eta_{p,1}) \cdot (r - s_n)}$$

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- The numerator is non-negative by inequality 4. The denominator is negative, which we can prove by combining inequalities of $\mathbb{S}$ labeled (2) and (3).

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- The numerator is non-negative by inequality 4. The denominator is negative, which we can prove by combining inequalities of $\mathbb{S}$ labeled (2) and (3).
- Thus, $E(\Delta\phi + cost_{R--LINE}) = v(G) \leq 0$.

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Thus, by properties (1), (2), (3), and (4), if $\mathbb{S}$ is satisfied then R–LINE is C-competitive.